1.29.09

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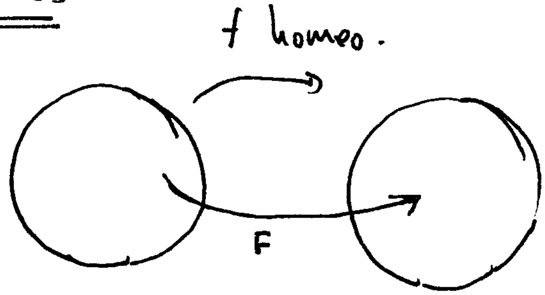


Question: "good" extension of f.

Conformally natural F.

$\gamma \in \mathrm{Conf}(S^{n-1}) \Rightarrow f \circ \gamma = \gamma \circ f$

$\Downarrow$

$F \circ \gamma = \gamma \circ F$

Conformal: (i) preserves the angles

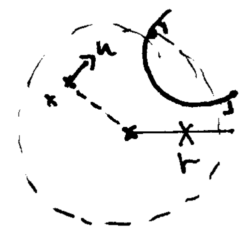
(ii) $n=3$, $S^2 \simeq \mathbb{C} \cup \{\infty\}$

$$z \to \frac{az+b}{cz+d}, \quad ad-bc \neq 0.$$

This question is related to Hyperbolic geometry.

In fact, $\mathrm{Conf}(S^{n-1}) = \mathrm{Iso}(\mathbb{H}^n)$.

Basics of hyperbolic geometry

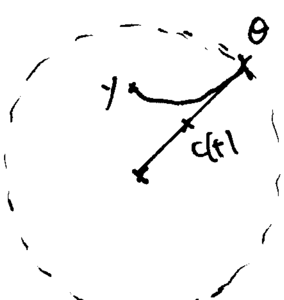


$$\|u\|_{hyp} = \frac{2\|u\|_{Euc}}{1-\|x\|^2}$$

$$d = d_{hyp}(0,r) = \log\left(\frac{1+r}{1-r}\right) \Leftrightarrow \tanh(d/2) = r.$$

Check: $\mathrm{Iso}(\mathbb{H}^n) \simeq \mathrm{Conf}(S^{n-1}) \simeq PO(1,n)$.

Busemann function



$$\lim_{t\to\infty} \left(d(y, c(t)) - t\right) = B_y(\theta).$$

$$B(0,\theta) = 0.$$

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Properties of Busemann

(i) $\forall \gamma \in \mathrm{Iso}(\mathbb{H}^n)$ we have $B(\gamma y, \gamma\theta) = B(y,\theta) + B(\gamma^{-1}(0), \theta)$.

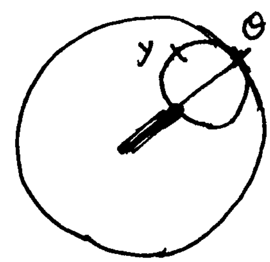
(ii) $|\nabla B| \equiv 1$

(iii) B are convex functions. $\quad t \mapsto B(\gamma(t), \theta)$ is convex.

(a) strictly convex if $\theta \neq \gamma(+\infty)$ and $\gamma(-\infty)$

(b) linear otherwise.

(iv) level sets $\rightsquigarrow$ spheres tangent to $S^{n-1} = \partial\mathbb{D}$.



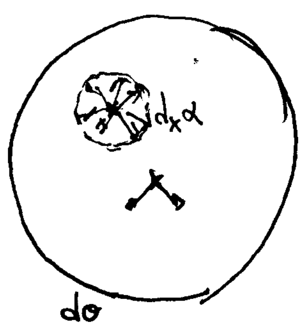
Wish to construct:

$$\mathbb{H}^n \to \mathcal{M}_1(S^{n-1})$$

$$x \to \mu_x$$

• embedding

$\begin{cases} \bullet\ \gamma \in \mathrm{Iso}(\mathbb{H}^n) \\ \mu_{\gamma x} = \gamma_*(\mu_x) \end{cases}$



$\mu_x =$ push forward of $d_x\alpha$.

$$\boxed{\frac{d\mu_x}{d\theta} = \exp\left(-(n-1)\, B(x,\theta)\right)}$$

<u>Example</u>: If $n=2$, $p(x,\theta) = \exp\left(B(x,\theta)\right)$

Euclidean Poisson Kernel.

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<u>Barycenter map:</u>

$\nu \in \mathcal{M}_1(S^{n-1})$. Take $\displaystyle\int_{S^{n-1}} B(x,\theta)\, d\nu(\theta)$.

average weighted distance from x to S^{n-1}.

<u>Note:</u> If ν has no atoms. $\qquad \nu = \frac{1}{2}\delta_{\theta_+} + \frac{1}{2}\delta_{\theta_-}$

$\mathcal{B}(x) = \frac{1}{2} B(x,\theta_+) + \frac{1}{2} B(x,\theta_-)$

<u>Lemma:</u> (i) $\mathcal{B}$ is strictly convex.

(ii) $\mathcal{B}(x) \xrightarrow[x \to S^{n-1}]{} +\infty$

<u>Corollary</u> $\exists !$ minimum $\rightsquigarrow$ bar(ν).

By construction $\forall \gamma \in \mathrm{Iso}(\mathbb{H}^n)$ $\quad \mathrm{bar}(\gamma_* \nu) = \gamma\, \mathrm{bar}(\nu)$.

$\boxed{\text{Define: } F(x) = \mathrm{bar}(f_*(\mu_x)).}$

<u>Lemma:</u> $$\boxed{\int dB_{(F(x),\theta)}\, d(f_*\mu_x) \equiv 0 \equiv \int_{S^{n-1}} dB_{(F(x),f(\theta))}\, d\mu_x .}$$

<u>Theorem:</u> (i) Douady-Earle, if $n=2$, F is an analytic diff.

(ii) B-Courtois-Gallot if $n \geq 3$, F is smooth

$|J\, F(x)| \leq 1$

(4)

Moreover, $f\circ\gamma = \partial\circ\gamma \Rightarrow F_\bullet\circ\gamma = \partial\circ F$.

<u>Proof:</u>

$$0 = \int_{S^{n-1}} dB_{(F(x),\, f(\theta))}\, e^{-(n-1)B(x,\theta)}\, d\theta$$

$u \in T_x \mathbb{H}^n \qquad v \in T_{F(x)} \mathbb{H}^n$.

$$\Rightarrow \int_{S^{n-1}} D\, dB_{0\,(F(x),\, f(\theta))} (dF(u), v)\, e^{-(n-1)B(x,\theta)}\, d\theta =$$

$$(n-1)\int dB_{(x,\theta)}(u)\; dB_{0\,(F(x),\, f(\theta))}(v)\; e^{-(n-1)B(x,\theta)}\, d\theta$$

$$\leq (n-1)\left(\int dB^2(u)\right)^{1/2}\left(\int dB_0(v)\right)^{1/2}$$

<u>Write:</u> $\langle K(v), v\rangle = \int D\, dB_0(v,v)$; $\langle H(v), v\rangle = \int dB_0^2(v)$

$\langle H'(u), u\rangle = \int dB^2(u)$.

$$\langle K\, d_xF(u), v\rangle \leq (n-1)\langle Hu, u\rangle^{1/2}\langle H'(v), v\rangle^{1/2}$$

<u>Remark:</u> trace (H) = trace $(H') = 1$

$(e_i)\quad \sum \langle H e_i, e_i\rangle = \int \sum^n dB_0(e_i)^2 = \int |dB_0|...$

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algebra $\Rightarrow$ $(\det K)\,|\mathrm{Jac}\,F(x)| \leq (n-1)^n\,(\det(H))^{1/2}\left(\frac{\mathrm{trace}\,H'}{n}\right)^{n/2}$

$$\leq \frac{(n-1)^n}{n^{n/2}}\,(\det(H))^{1/2}.$$

$$|\mathrm{Jac}\,(F(x))| \leq \frac{(n-1)^n}{n^{n/2}}\,\frac{(\det(H))^{1/2}}{\det(K)} = \frac{(n-1)^n}{n^{n/2}}\,\frac{(\det H)^{1/2}}{\det(Id-H)}$$

$\boxed{n\geq 3}$

<u>Fact:</u> $D\,dB_0(v,v) = \langle v,v\rangle - (dB_0(v))^2$

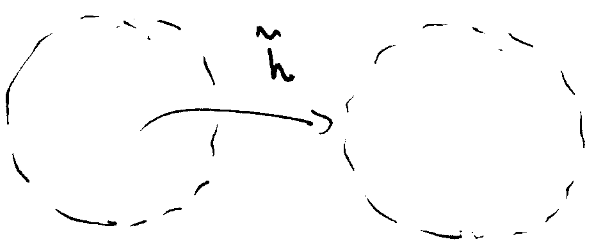
Take $K = Id - H$.

<u>Lemma:</u> $\mathcal{F}(H) = \frac{\det(H)^{1/2}}{\det(I-H)} \Rightarrow \mathcal{F}(H) \leq \mathcal{F}(\frac{1}{n}Id)$

<u>Mostow Rigidity</u>

$\mathbb{H}^n/\Gamma_1 \xrightarrow[\simeq]{h} \mathbb{H}^n/\Gamma_2$ $\qquad \rho:\Gamma_1\to\Gamma_2$

compact manifolds with $\Gamma_1 \simeq \Gamma_2 \Leftrightarrow h$ conti. map which is homotopic to id.



. $\forall\gamma\in\Gamma_1$, $\tilde{h}(\gamma x) = \rho(\gamma)\,\tilde{h}(x)$

. $\tilde{h}$ extends to a homeo. of S^{n-1}.

$\downarrow$

F from previous construction

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$\Rightarrow \mathrm{vol}(\mathbb{H}^3/\Gamma_2) = \mathrm{vol}(\mathbb{H}^3/\Gamma_1) \Rightarrow F$ is an isometry.