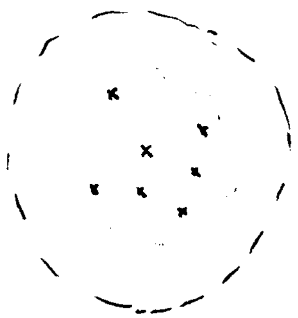
1.30.09 Geometrical Problem Gilles Courtois

$\Gamma_{\text{discrete}} \subset \mathrm{Iso}(X)$ simply connected manifold of sectional $K \leq -1$.

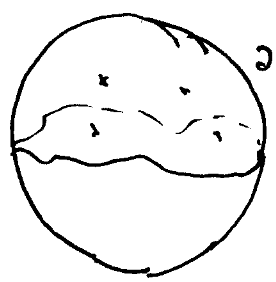
size of Γ:



$$\delta(\Gamma) = \lim_{R \to \infty} \frac{1}{R} \log \#\{\gamma o \mid d(\gamma o, o) \leq R\}$$

$$= \inf\{c > 0 \mid \sum_{\gamma \in} e^{-c d(x, \gamma x)} < \infty\}.$$

$\Lambda(\Gamma) = \overline{\Gamma o} \cap \partial X$

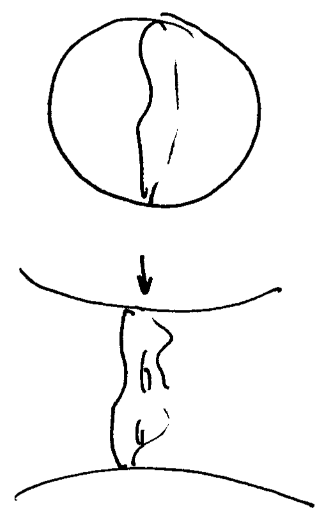


∂X = boundary

Theorem: (Besson, G, Gallot).

$H_k(X/\Gamma, \mathbb{R}) \neq 0$ for some $k \geq 2 \Rightarrow \delta(\Gamma) \geq k-1$.

$\mathrm{inj}(X/\Gamma) > 0$.



Intuition

Take $0 \neq Z \in H_n(X/\Gamma, \mathbb{R})$.

Proof: Build map $f_c : X/\Gamma \to X/\Gamma$,

f_c homotopic to the identity, $\forall\, Z \subset H_k(X/\Gamma, \mathbb{R})$

$$\mathrm{vol}(f(Z)) \leq (\delta(\Gamma) + \eta)^k \,\mathrm{vol}(Z).$$

②

Note: $f^{(n)} Z \sim Z$. If $\delta(\Gamma) < k-1$,

$$\mathrm{vol}\, f^{n}(Z) \leq \left(\frac{\delta(\Gamma)+1}{k}\right)^{nk} \mathrm{vol}(Z) \to 0.$$

Construction of a family of maps f_c:

$c \in \mathbb{R}$, $x, y \in X$ define:

$$P_c(x,y) = \sum_{\gamma \in \Gamma} e^{-c\rho(\gamma o, x)} \cosh \rho(\gamma o, y), \quad o \text{ fixed point of } X$$

ρ is a distance on X.

Fact: (1) $P_c(x,y) < \infty$ $\forall c > \delta(\Gamma)+1$.

Then $P_c(o,o) = \sum_{\gamma \in \Gamma} e^{-(c-1)\rho(\gamma o, o)} < \infty$ for $c-1 > \delta(\Gamma)$

(2) $P_c(\gamma x, \gamma y) = P_c(x,y)$ $\forall x, y \in X$, $\forall \gamma \in \Gamma$

(3) $\forall x$, $P_c(x,y) \xrightarrow[y \to \partial X]{} +\infty$

(4) $\forall x$, $y \longmapsto P_c(x,y)$ is strictly convex

$K \leq -1 \Rightarrow Dd\rho \geq$

distance function is more convex than $K=-1$

Define: $f_c(x) = y$ minimizer of $P_c(x, \cdot)$

③

$\forall c > \delta(r)+1$

<u>Propo</u> $\cdot \, \| df_c(x) \| \leq c$

$\cdot \, \left| \mathrm{Jac}_k \, f_c(x) \right| \leq \left(\frac{\delta(r)+1}{k} \right)^k \qquad k \geq 2$

$(u_1, \dots, u_n)$ o.n. frame in $T_x X$

$$\left\| \sup_{\substack{(u_1,\dots u_n) \\ \text{orthonormal}}} \| df_c(u_1) \wedge \dots df_c(u_n) \| \right.$$

<u>Lehmer:</u> Let $\pi \in \mathbb{Z}[X]$ irreducible polynomial.

$$\pi(x) = X^n + a_{n-1} X^{n-1} + \dots + a_1 x + a_0$$

$(x_i)_{i=1}^{n}$ roots of $\pi \in \mathbb{C}$.

<u>Mahler Norm</u> $m(\pi) = \prod_{i=1}^{n} \max(1, |x_i|)$.

<u>Lehmer's Question</u> $\exists c > 1, \ (\forall \pi). \ m(\pi) > 1 \Rightarrow m(\pi) > c$.

<u>Proved:</u>

- odd degree polynomials, P. Borwein, ...

- non palindromic polynomials. ($x^n \pi(1/x) = \pi(x)$, $n = \deg \pi$).

$r^2 / n \log(1 + n/r)$

④

Conjecture Lehmer $\Rightarrow$ Margulis Conj.

G semi-simple Lie group of $\mathbb{R}$-rank ≥ 2

$\exists\, e \in U \subset G$, $\forall \Gamma$ co-compact lattice in G, $\Gamma \cap U$ contains only elliptic elements

x x x

π : $\pi(x) = 0$.

<u>Consider:</u> $G = \left\langle \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} x & 1 \\ 0 & 1 \end{pmatrix} \right\rangle \subset GL(2, \mathbb{C})$.

$$= \left\{ \begin{pmatrix} x^k & p(x) \\ 0 & 1 \end{pmatrix} \;\middle|\; \begin{matrix} k \in \mathbb{Z} \\ p \in \mathbb{Z}[x] \\ \deg(p) \leq n-1 \end{matrix} \right\} \cong \mathbb{Z} \ltimes_A \mathbb{Z}^n$$

$$A = \begin{pmatrix} 0 & \cdots & -a_0 \\ 1 & & -a_1 \\ \vdots & & \vdots \\ 0 & & 1-a_{n-1} \end{pmatrix}$$

<u>Claim:</u> $G \xrightarrow{\sim} \Gamma_{\text{discrete}} \quad SL(2,\mathbb{R}) \times \cdots \times SL(2,\mathbb{R})$

$$\begin{pmatrix} x^k & p(x) \\ 0 & 1 \end{pmatrix} \mapsto \left[\begin{pmatrix} 1 & p(x_1) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1^{k/2} & 0 \\ 0 & x^{k/2} \end{pmatrix}, \cdots \cdots \right]$$

Fact: $\Gamma \subset \mathbb{H}^2 \times \cdots \times \mathbb{H}^2 = X$ and if π is ~~non~~ palindromic

5

$\exists f_c : X/\!\!\!/\,\mu \to X/\!/\mu.$

Theorem: $\delta(\mu) \geq 3-1=2.$

Question: relation between $\delta(\mu)$ and $m(\pi)$.

$$G = \left\langle \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} x & 1 \\ 0 & 1 \end{pmatrix} \right\rangle$$

$\text{entropy}(G) \leq \log m(\pi).$