

MATH 3500(H)
PROBLEM SET #12

DUE Wednesday, November 9, 2011.

Problems to work but not hand in:

§5.1: #1, 3, 4a, 9.

§5.2: #1a,i, 3.

Problems to turn in:

No WeBWork assignment this week.

§5.1: #2* (3), 4b (3), 8 (3), 10 (3), 11[†] (3), 12 (3).

A. (3) Suppose $A \subset \mathbb{R}^n$ is closed but unbounded, $f: A \rightarrow \mathbb{R}$ is continuous, and $f(\mathbf{x}) \rightarrow 0$ as $\|\mathbf{x}\| \rightarrow \infty$ with $\mathbf{x} \in A$.[‡] Prove that if $f(\mathbf{x}) > 0$ for all $\mathbf{x} \in A$, then there is $\mathbf{z} \in A$ so that $f(\mathbf{x}) \leq f(\mathbf{z})$ for all $\mathbf{x} \in A$.

Hint: Pick an arbitrary $\mathbf{a} \in A$ and, referring to the footnote, take $\varepsilon = f(\mathbf{a})$. Using this, define a compact set $X \subset A$ and apply Theorem 1.2. Last, argue that $f(\mathbf{z}) \leq f(\mathbf{x})$ for all $\mathbf{x} \in A$.

B. (3) Suppose $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ is differentiable and $f\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = f\left(\begin{smallmatrix} y \\ x \end{smallmatrix}\right)$ for all $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$.

(i) Show that $\begin{bmatrix} x \\ y \end{bmatrix}$ is a critical point of f if and only if $\begin{bmatrix} y \\ x \end{bmatrix}$ is also.[§]

(ii) Contrary to popular belief, there need not be critical points with $x = y$. Find *all* the critical points of $f\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = 2(x + y) + xy(1 + x + y)$.

§5.2: #4 (3), 5 (4).

Challenge problems (Turn in separately):

§5.1: #5 (3), 6 (2), 7 (3), 13 (4).

*Hint: See Exercise 2.3.2. Since there are two ways X can fail to be compact, you will need to give a function f for each of those cases.

[†]Hint: Choose $\mathbf{x}_k \in S_k$ for each $k \in \mathbb{N}$.

[‡]More precisely, this means that given any $\varepsilon > 0$, there is $R > 0$ so that whenever $\mathbf{x} \in A$ and $\|\mathbf{x}\| > R$, we have $|f(\mathbf{x})| < \varepsilon$.

[§]Hint: Chain rule.