Common Core K – 12 Math Standards and Teacher Education

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The Common Core State Standards Initiative

Governors and state commissioners of education from 48 states, 2 territories and the District of Columbia have committed to developing a common core of state standards in English-language arts and mathematics for grades K-12. Common Core State Standards Initiative (CCSSI) is a state-led effort coordinated by the National Governors Association Center for Best Practices (NGA Center) and the Council of Chief State School Officers (CCSSO).

Implications for teacher education

Obviously, teachers must understand the mathematics they will teach.

However, in my experience, many of the mathematical ideas in the CCSS are not well understood by prospective teachers when they begin their preparation program.
Developing a reasonable understanding of these ideas takes more time than one might guess (no matter how smart the students are).

Next, I will give a number of examples of mathematical ideas that are not obvious or previously known to prospective teachers, but that teachers will be expected to teach in the CCSS.

Kindergarten and before: Counting

The number word list versus counting how many

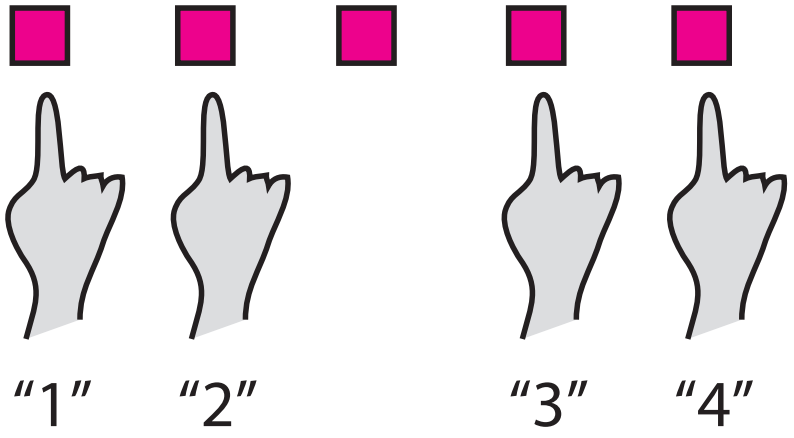
If a child can correctly say the first five counting numbers,

“one, two, three, four, five,”

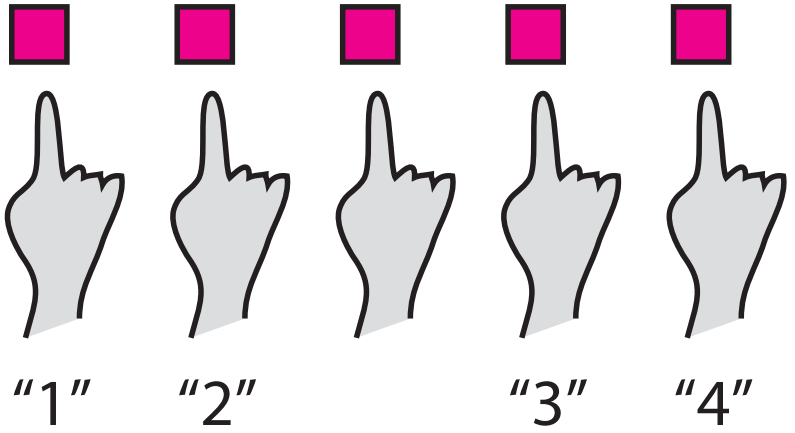
will the child necessarily be able to determine how many blocks there are in this collection?

One-to-one correspondence between number words and objects

Child 1:



Child 2:



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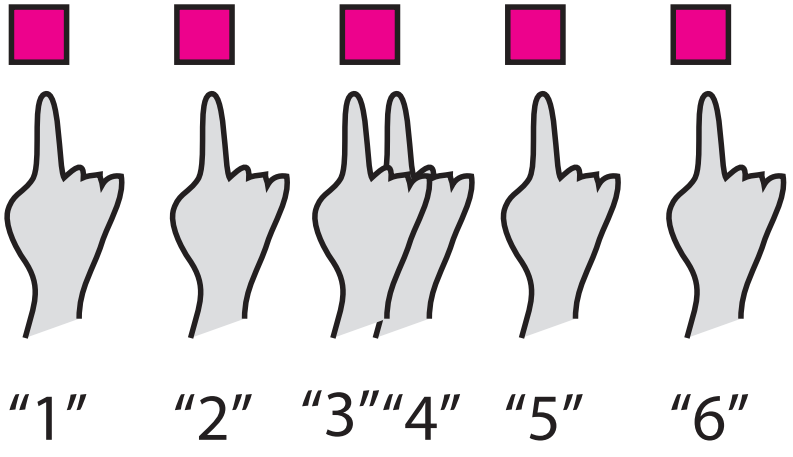

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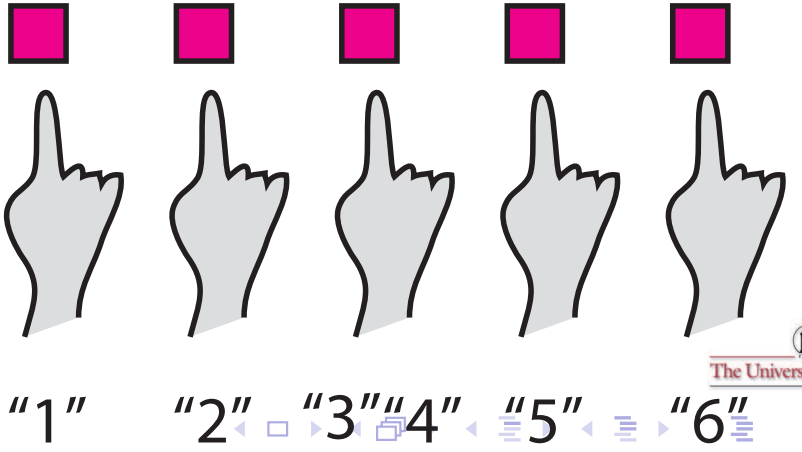

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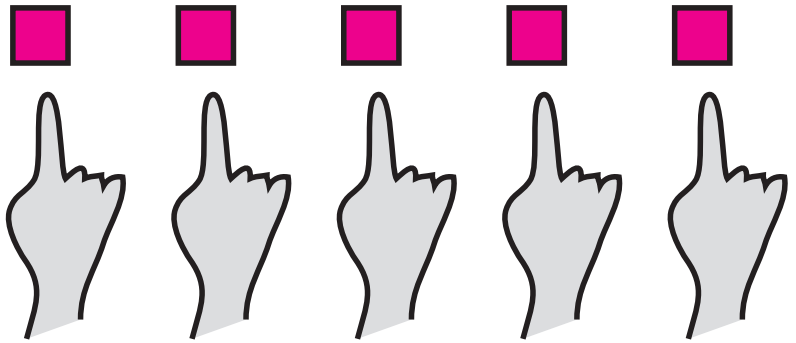
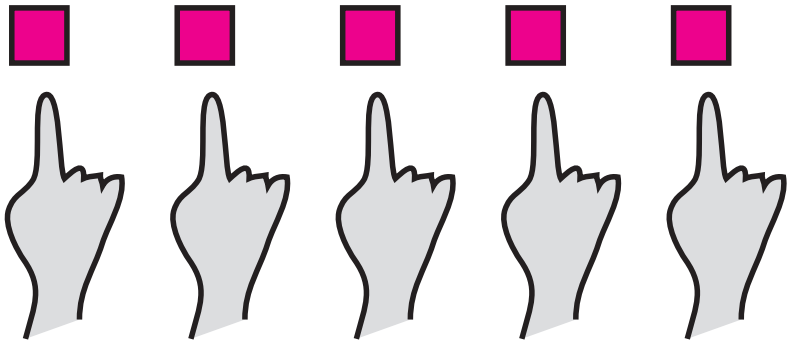
The last number word tells how many in all

Teacher: "How many blocks are there?"

Child 1:

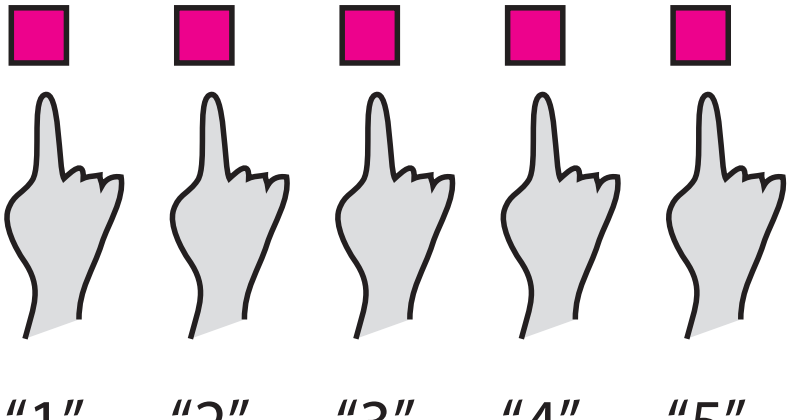


Child 2:

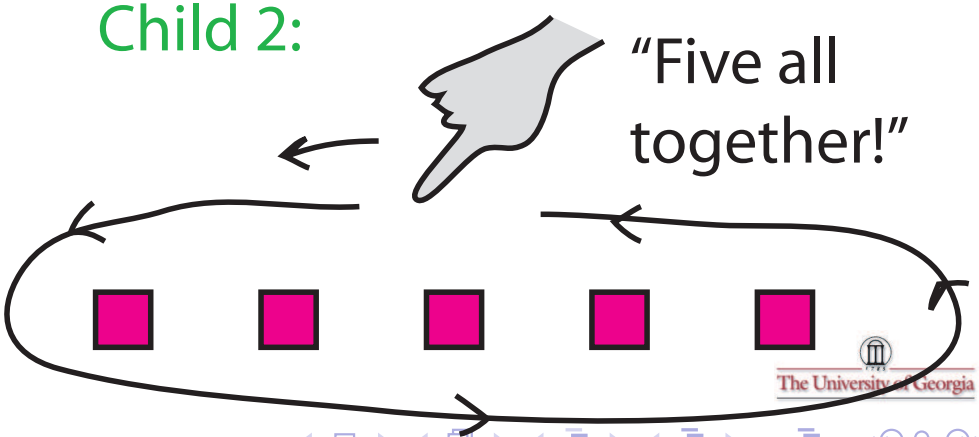


Teacher: "So how many blocks are there?"

Child 1:



Child 2:



Shifting between the number word list and how many

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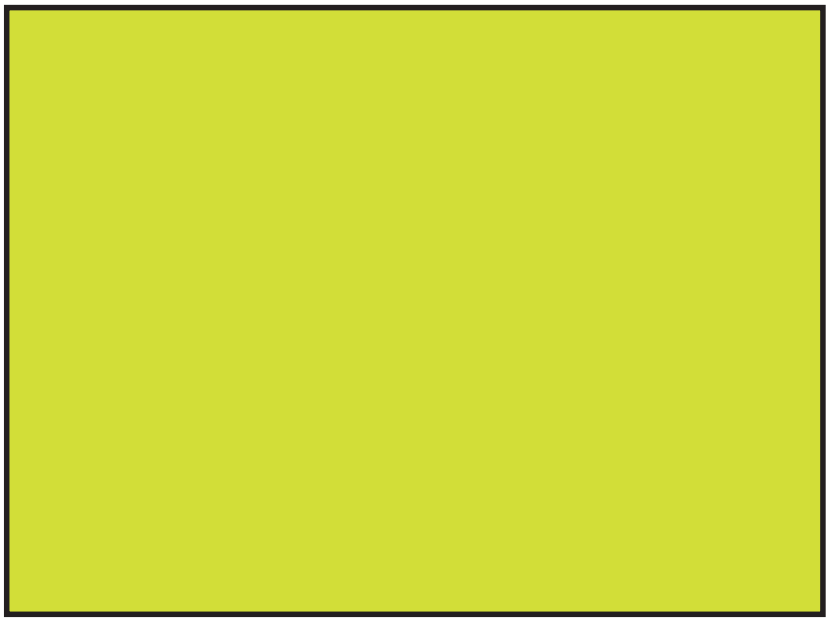
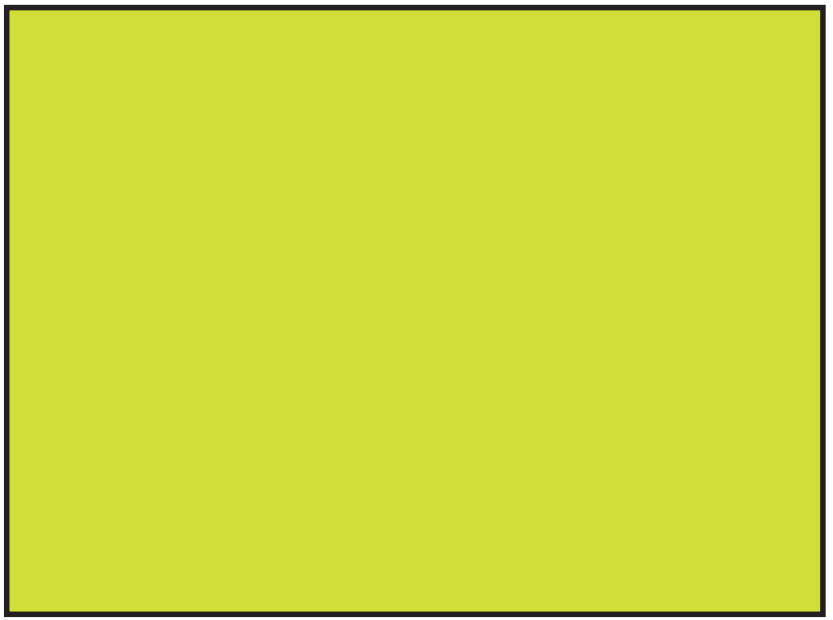


Hide them.

Shifting between the number word list and how many

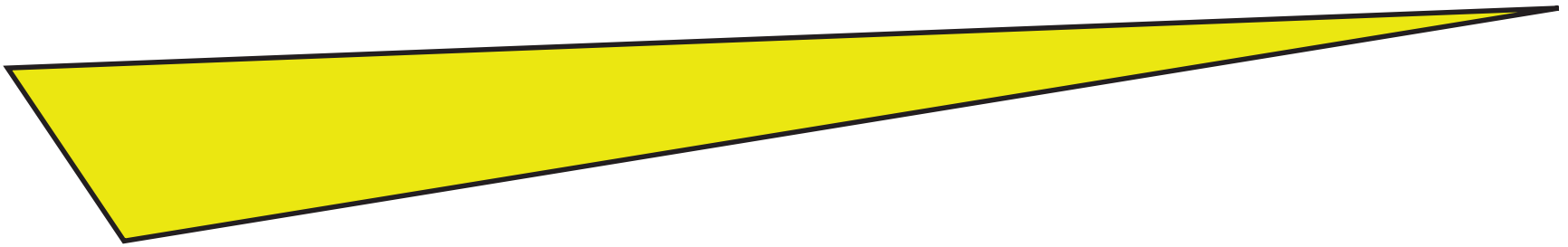


Ask: How many bugs are there altogether?

Kindergarten and before: shapes in different orientations

Recognizing triangles

“What is that pointy thing?”
“It’s a triangle!”
“That’s not a triangle! It doesn’t look like a triangle.”
“But it is a triangle. I know it is because it has 3 straight sides and 3 angles. So even though it looks different, it has to be a triangle because that’s what triangle means.”



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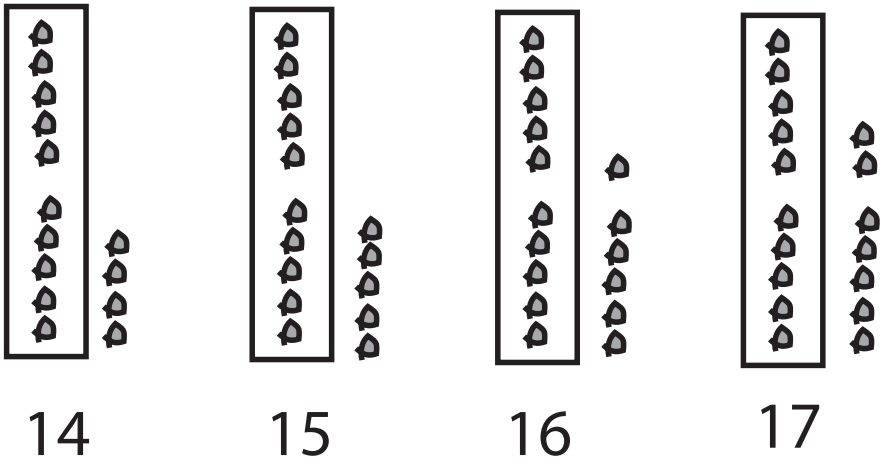
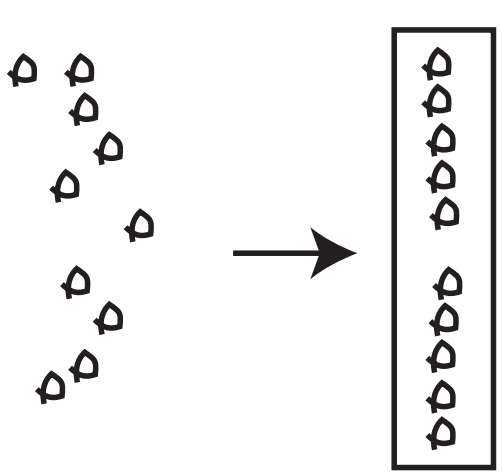
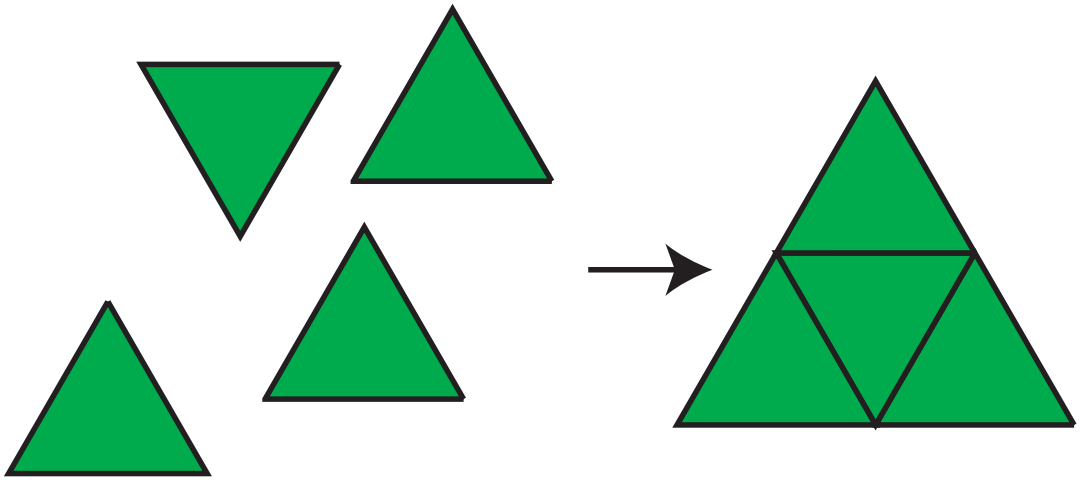

14
15
16
17

Unitizing: grouping to create a new unit





10 ones are grouped
to form one ten

Grade 1 and up: Addition and subtraction word problems

	Result unknown	Change unknown	Start unknown
Add to	$2 + 3 = ?$	$2 + ? = 3$	$? + 3 = 5$
Take from	$5 - 2 = ?$	$5 - ? = 3$	$? - 2 = 3$

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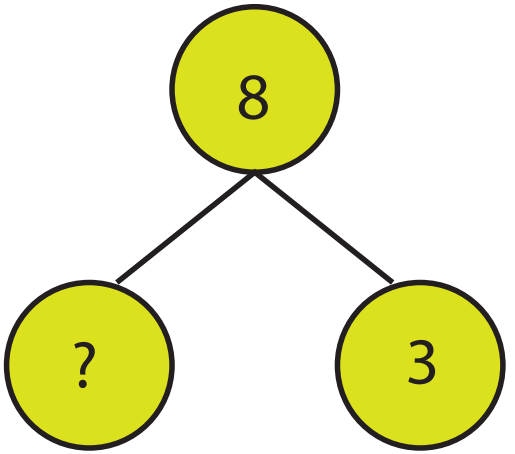
Addition and subtraction word problems

Use of keywords to solve is not reliable

Kwon has some cars. He gets 3 more cars. Now he has 8 cars in all. How many cars did Kwon have before he got more?

Add to, start unknown — students who rely only on keywords may mistakenly *add* 3 and 8.



Grade 1 and up: Addition and subtraction word problems

	Total unknown	Addend unknown	Both addends unknown
Put together/ Take apart	$3 + 2 = ?$	$3 + ? = 5$ $5 - 3 = ?$	$5 = 0 + 5$ $5 = 5 + 0$ $5 = 1 + 4$ $5 = 4 + 1$ $5 = 2 + 3$ $5 = 3 + 2$

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Grade 1 and up: Addition and subtraction word problems

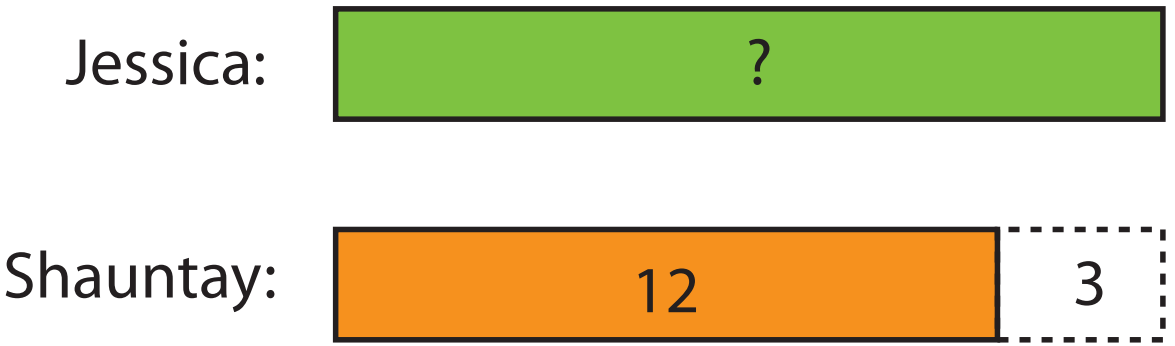
	Difference unknown	Bigger unknown	Smaller unknown
Compare	$2+?=5$ $5-2=?$	$2+3=?$ $3+2=?$	$5-3=?$ $?+3=5$

Addition and subtraction word problems

Use of keywords to solve is not reliable

Jessica has some cards. Shauntay has 3 fewer cards than Jessica. Shauntay has 12 cards. How many cards does Jessica have?

Compare, bigger unknown, “fewer” wording — students who rely only on keywords may mistakenly *subtract* 3 from 12.

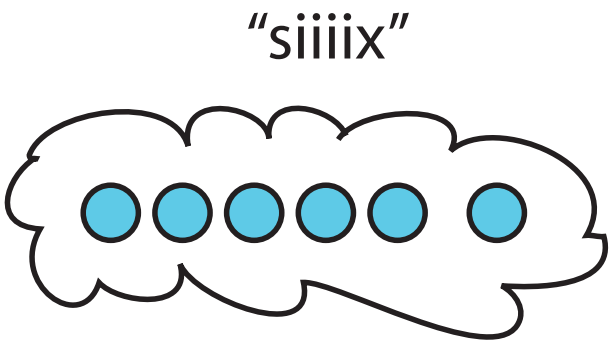


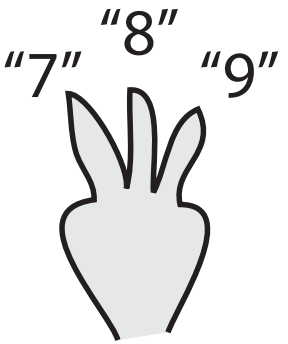
Grade 1: Addition and subtraction

Progression of numerical strategies for solving addition and subtraction problems (working toward fluency):

- Level 1: count all
- Level 2: count on, count on from larger, count on to subtract
- Level 3: derived fact methods, especially make-a-ten methods

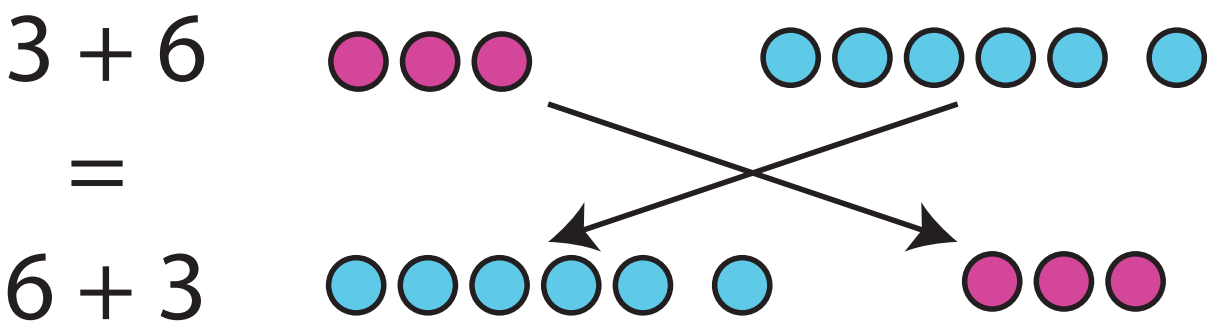
Level 2: Counting on

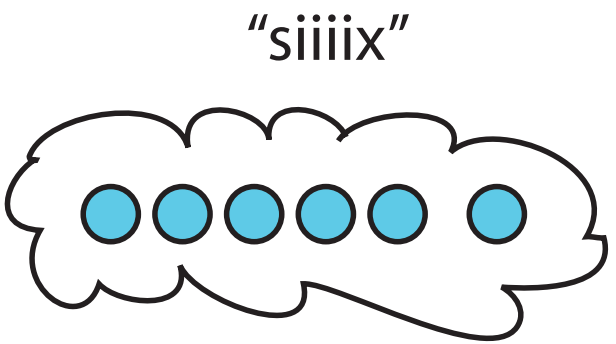


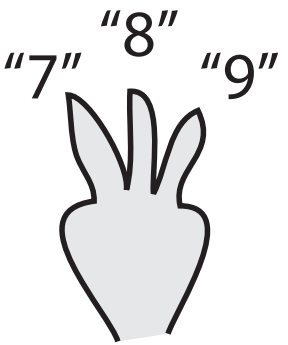


“so 6 + 3 = 9”

Level 2: Applying commutativity to count on from larger



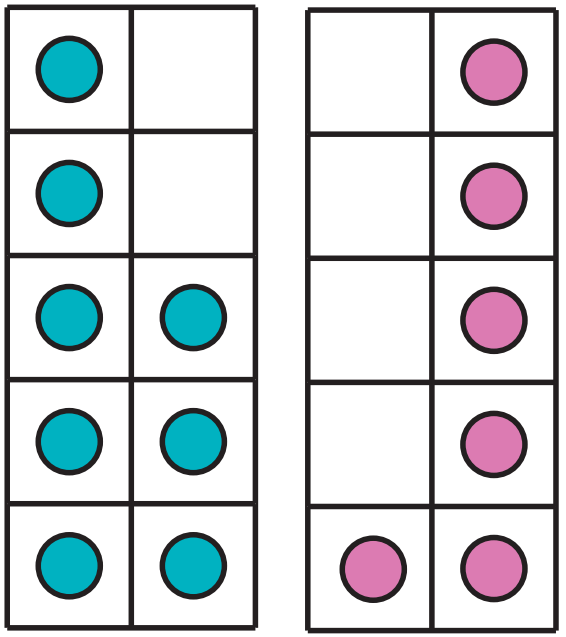




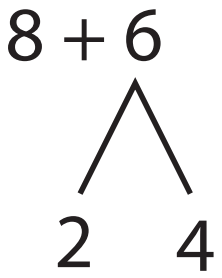
“so 6 + 3 = 9
3 + 6 = 9”

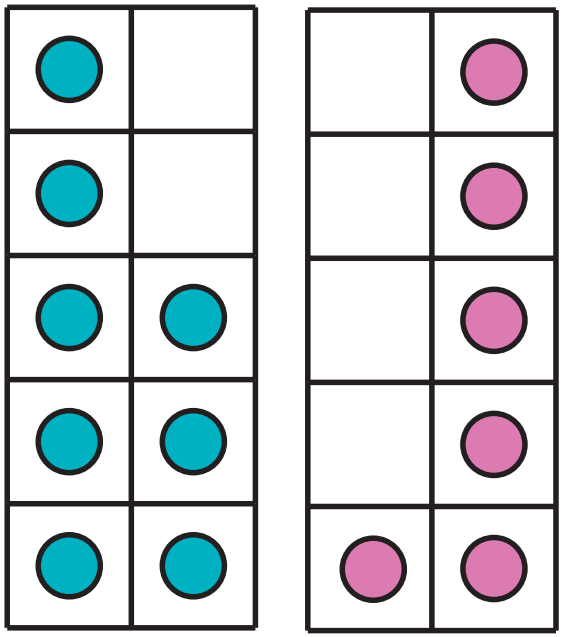
Level 3: Emphasizing grouping by tens

$8 + 6$



Level 3: Emphasizing grouping by tens

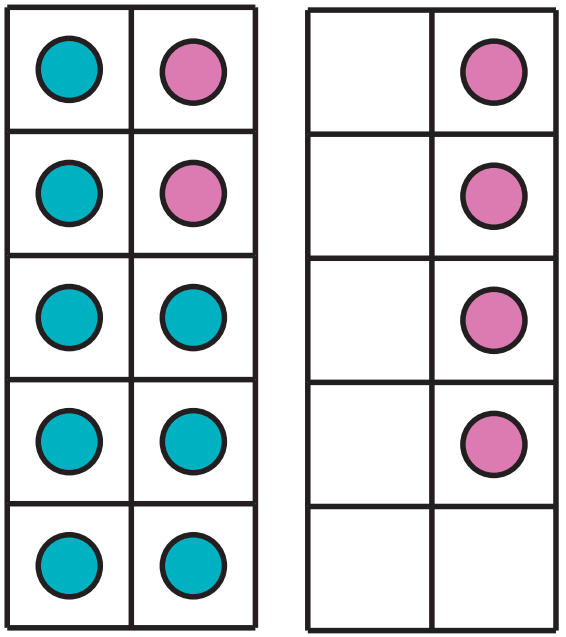




Level 3: Emphasizing grouping by tens

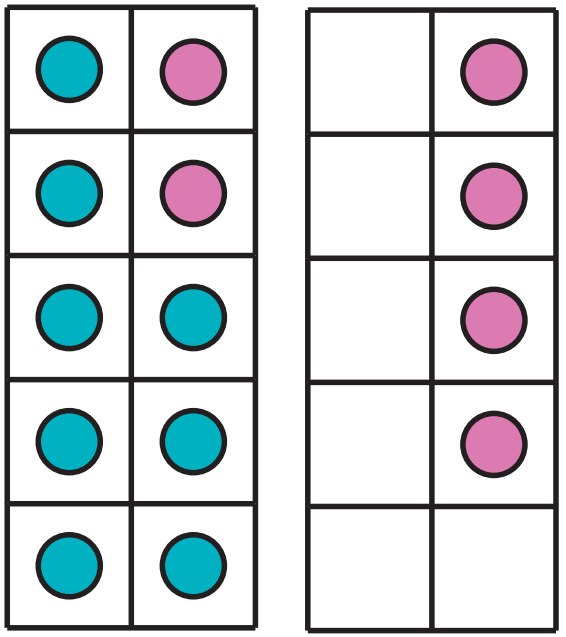
$$8 + 6 = 8 + (2 + 4)$$

2 4



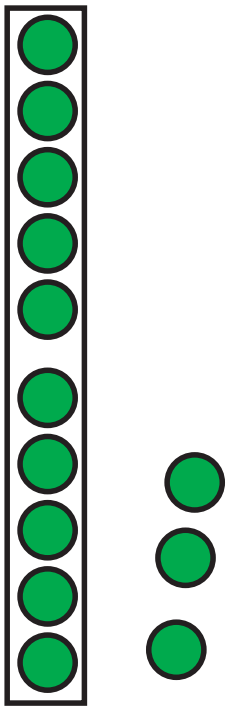
Level 3: Emphasizing grouping by tens

$$8 + 6 = 8 + (2 + 4) = (8 + 2) + 4 = 14$$

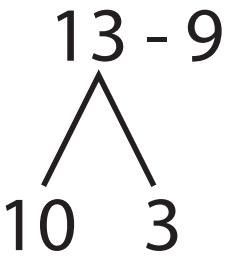


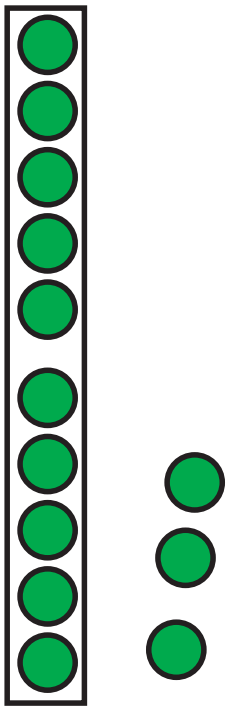
Level 3: Emphasizing grouping by tens

13 - 9

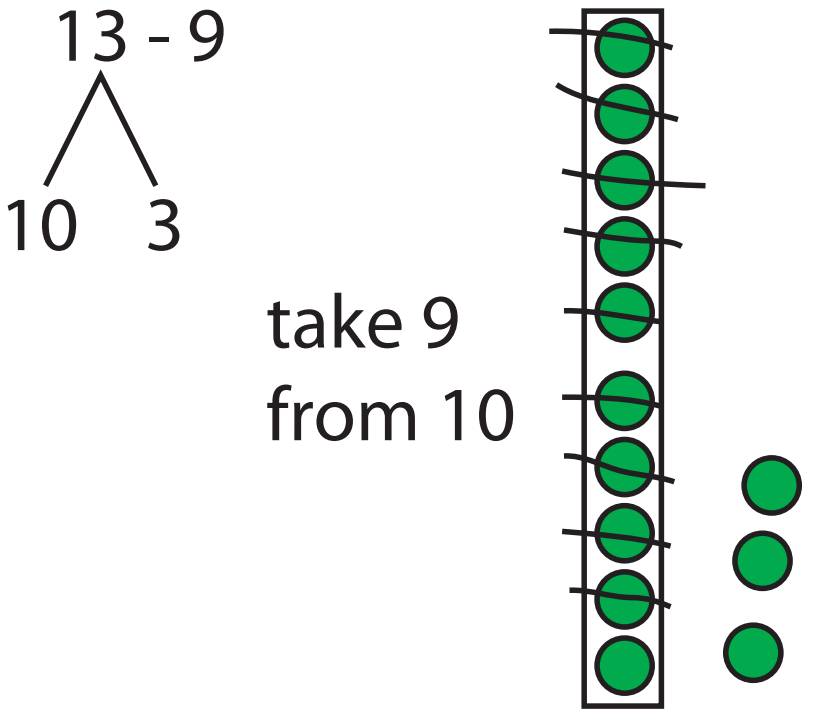


Level 3: Emphasizing grouping by tens

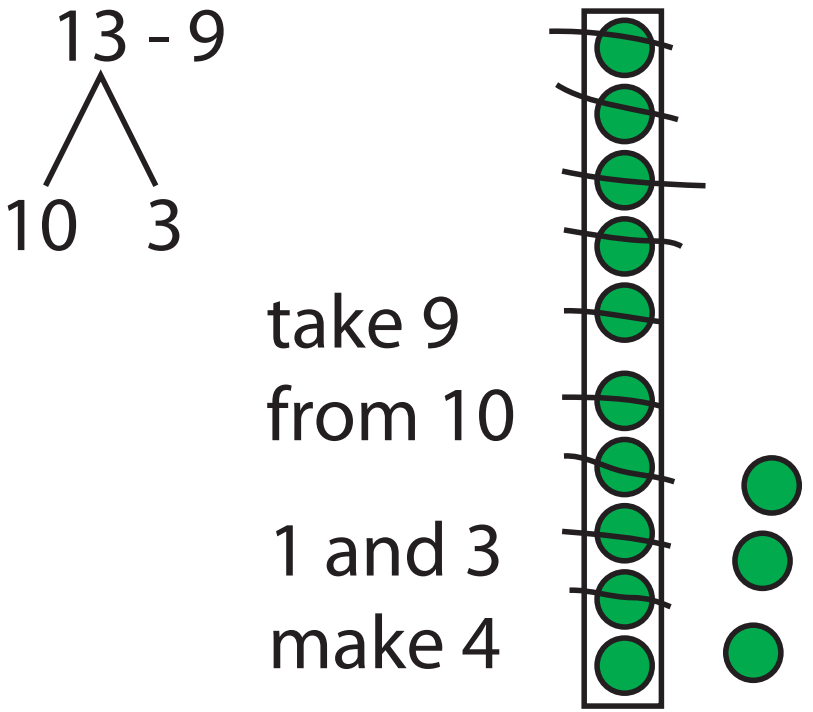




Level 3: Emphasizing grouping by tens

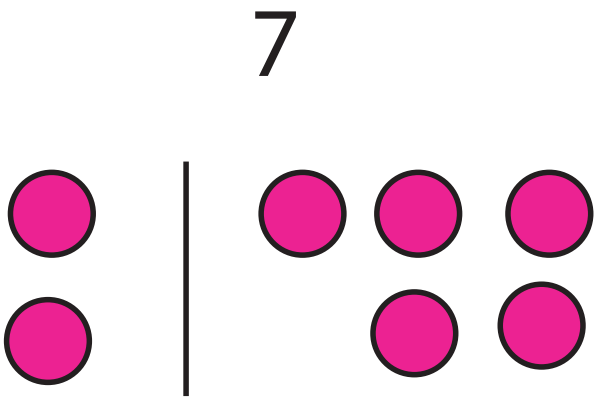


Level 3: Emphasizing grouping by tens

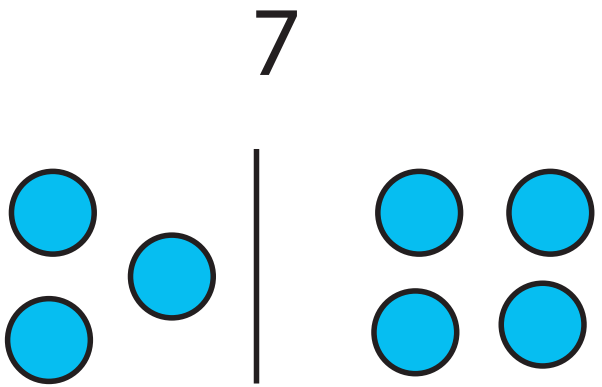


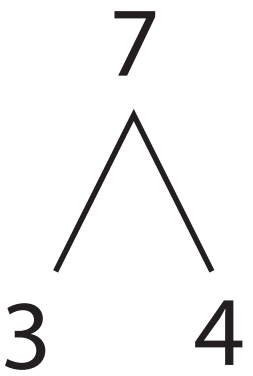
Grade 1: Composing and decomposing

Level 3 requires breaking numbers apart into partners



$7 = 2 + 5$

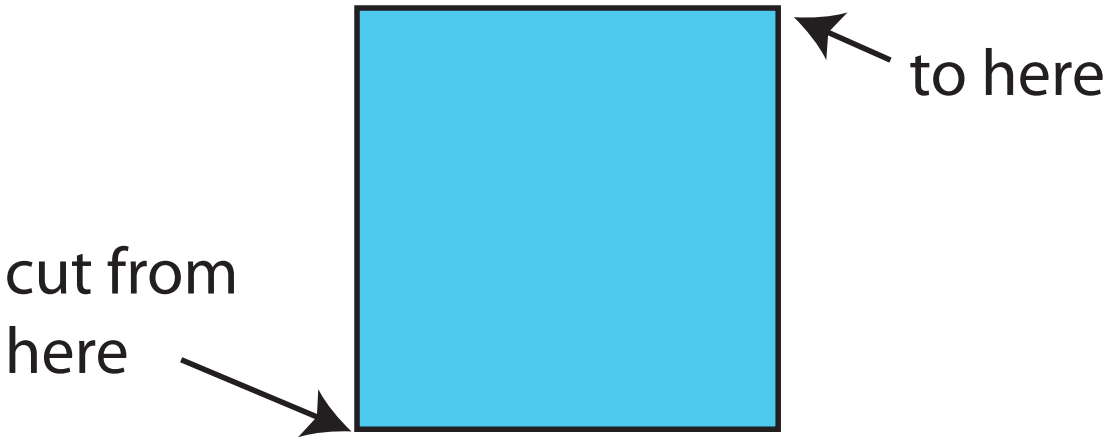




$7 = 3 + 4$

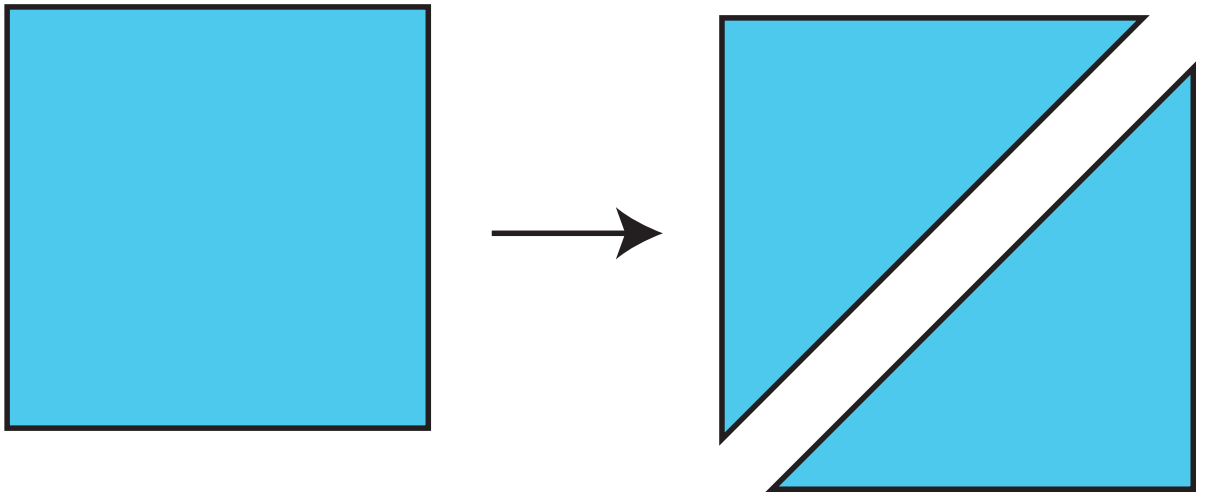
Decomposing a square and recomposing

A 1st grade teacher might ask:
"What if we cut the square from one corner to the opposite corner?
What shapes will we get?"



Decomposing a square and recomposing

"We get two triangles!"



Can we put the triangles together in other ways?

Decomposing a square and recomposing

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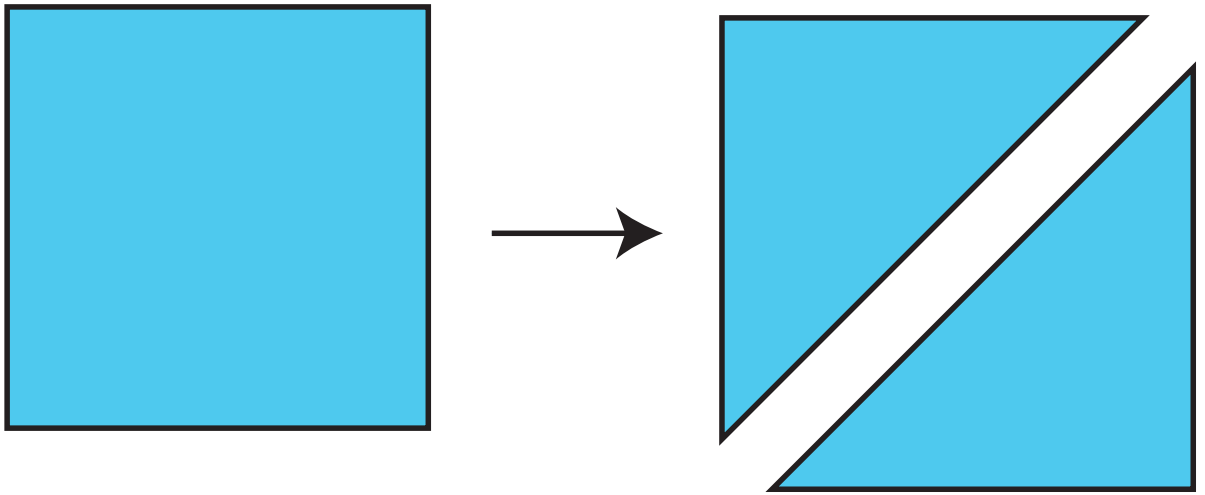

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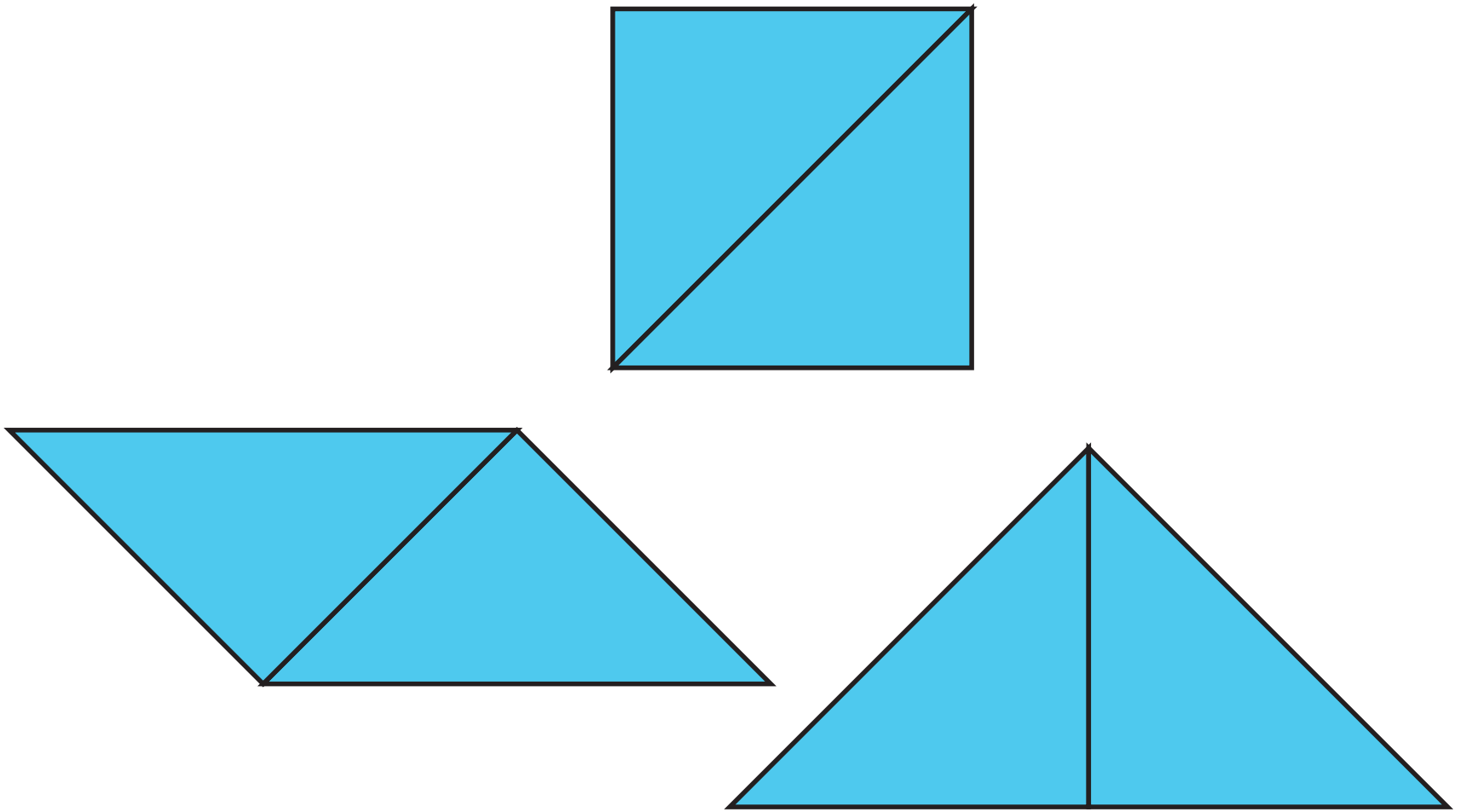
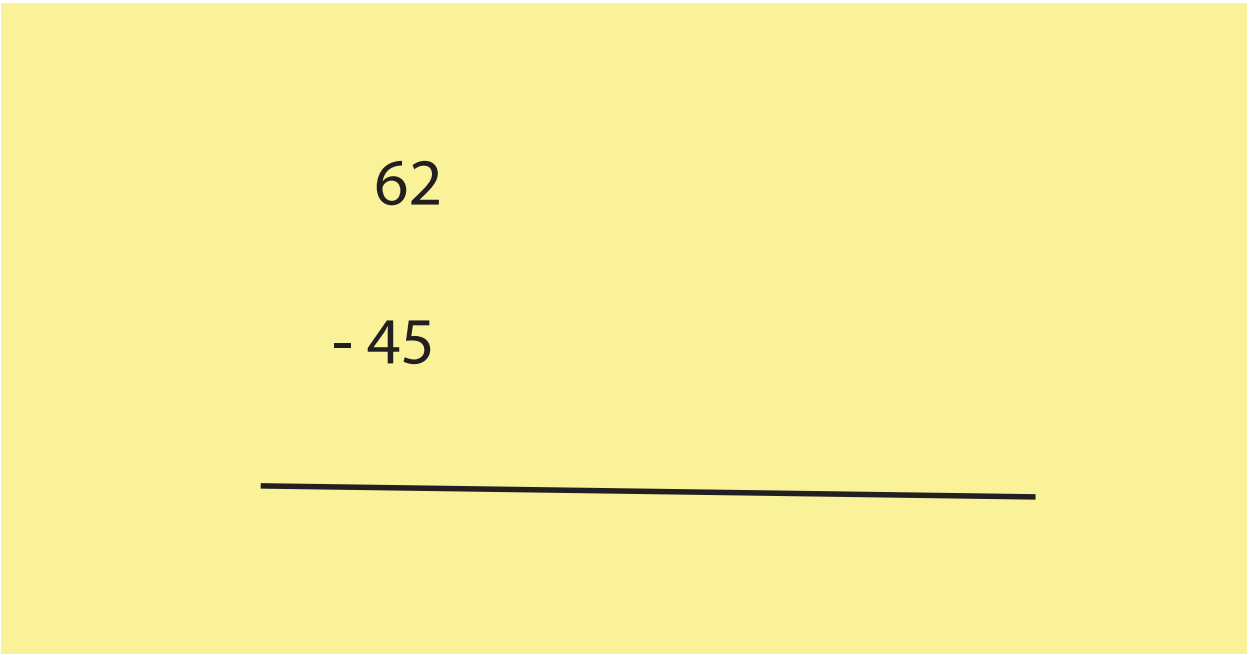


Grade 2

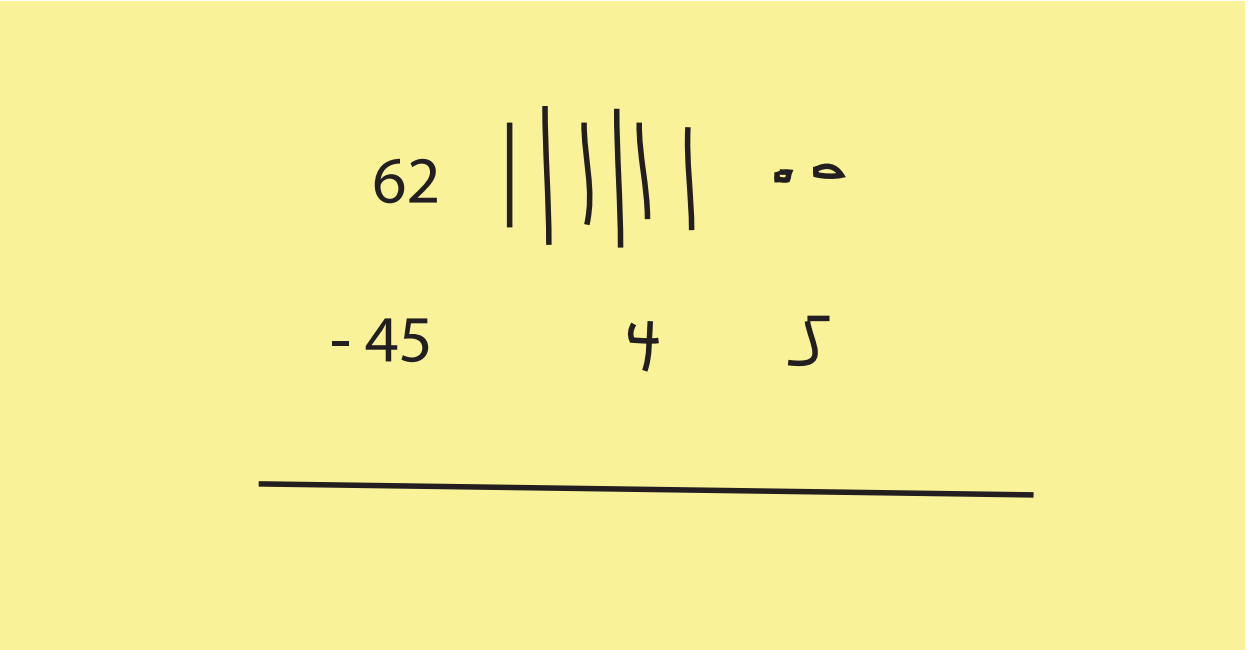
A grade 2 Common Core standard:

- Explain why addition and subtraction strategies and algorithms work, using place value and the properties of operations. Include explanations supported by drawings or objects. A range of reasonably efficient algorithms may be covered, not only the standard algorithm.

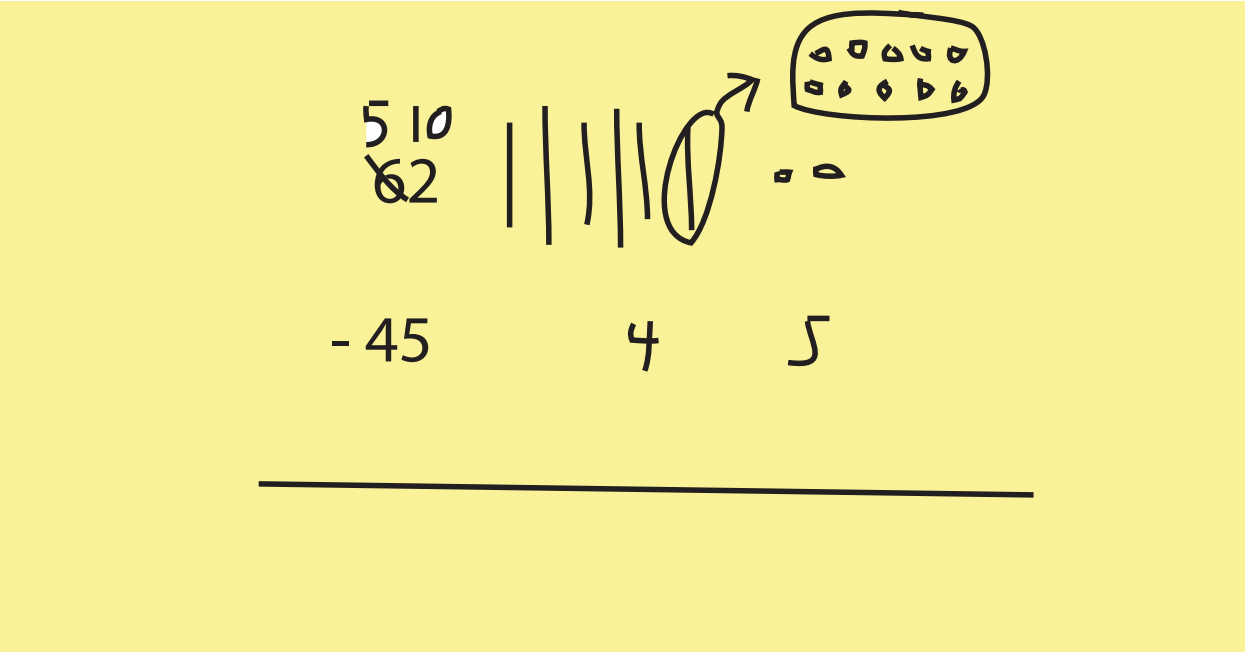
Understanding and explaining subtraction



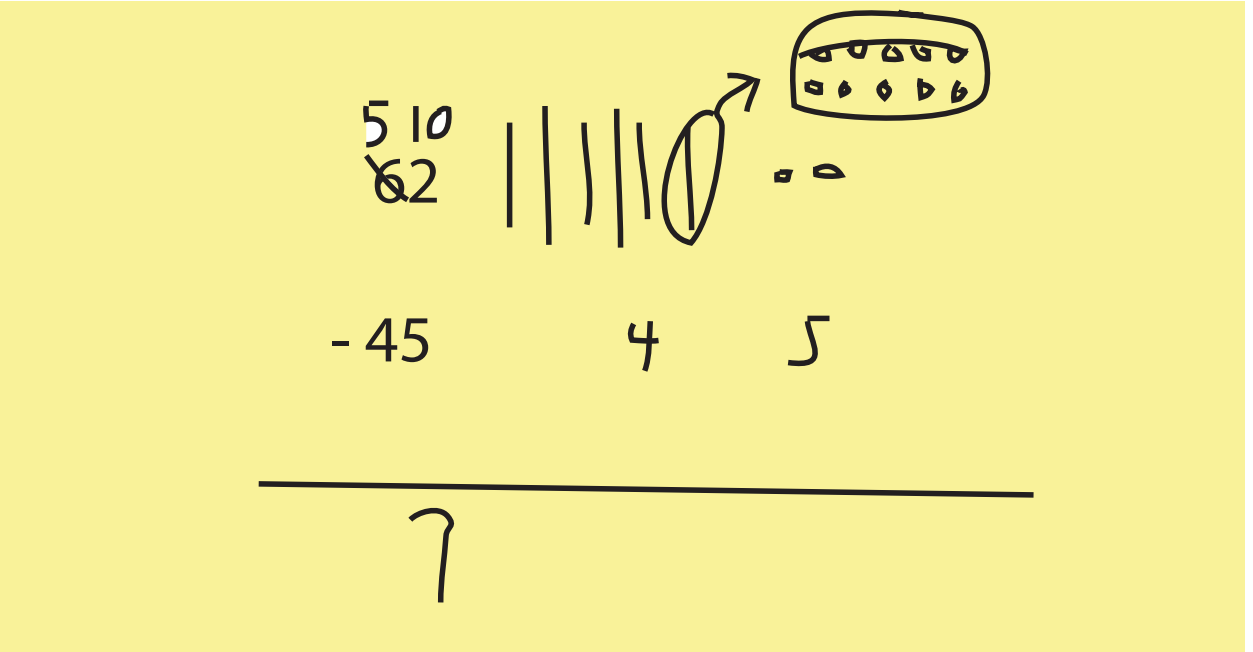
Understanding and explaining subtraction



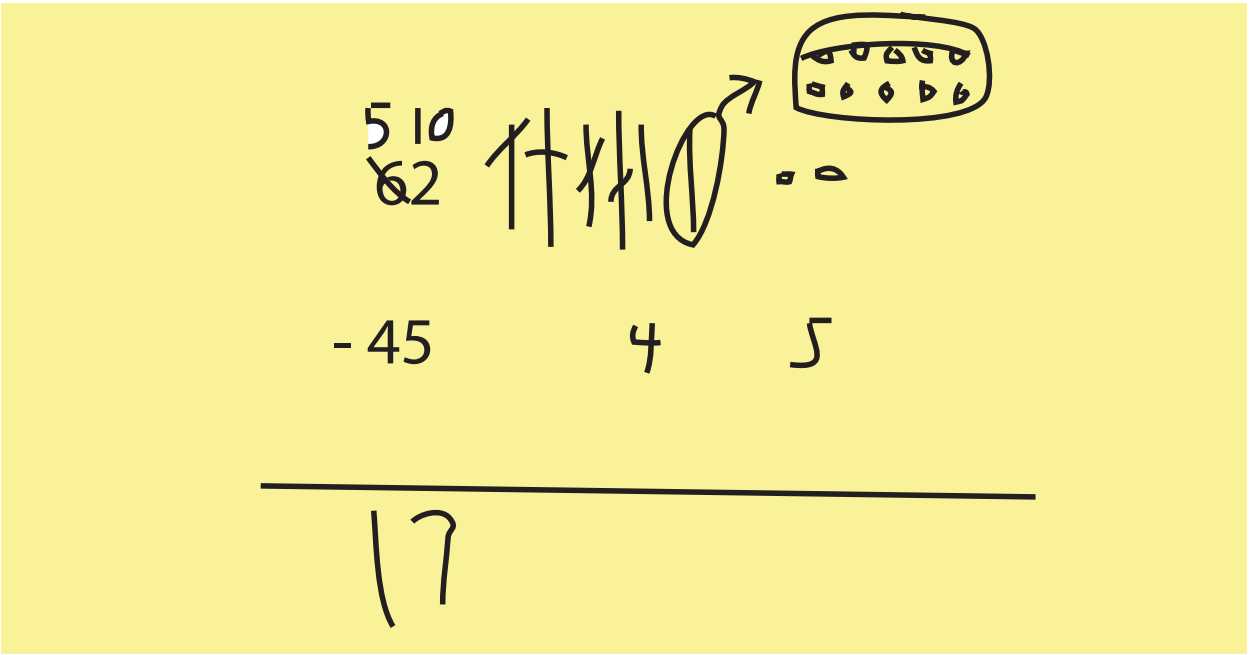
Understanding and explaining subtraction



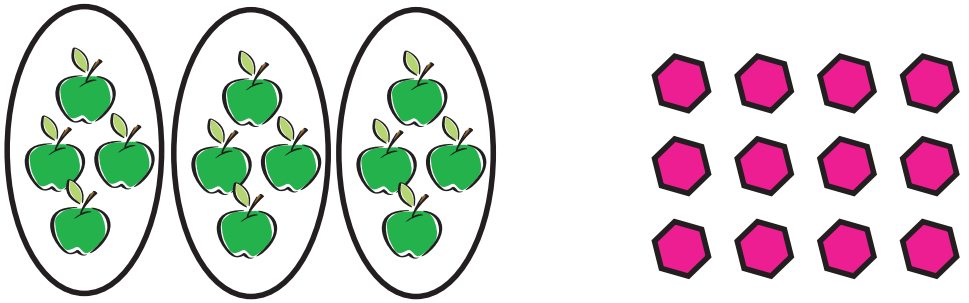
Understanding and explaining subtraction



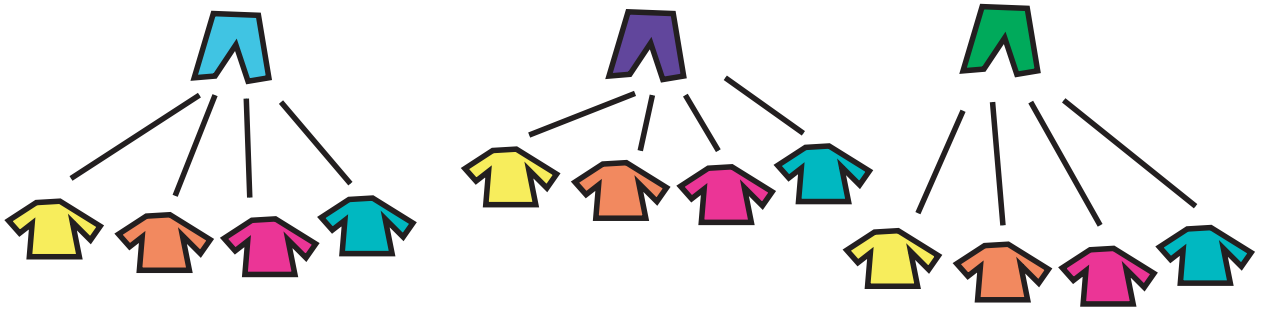
Understanding and explaining subtraction



Grade 3: Types of multiplication and division word problems



3×4



Grade 3: the commutative property of multiplication

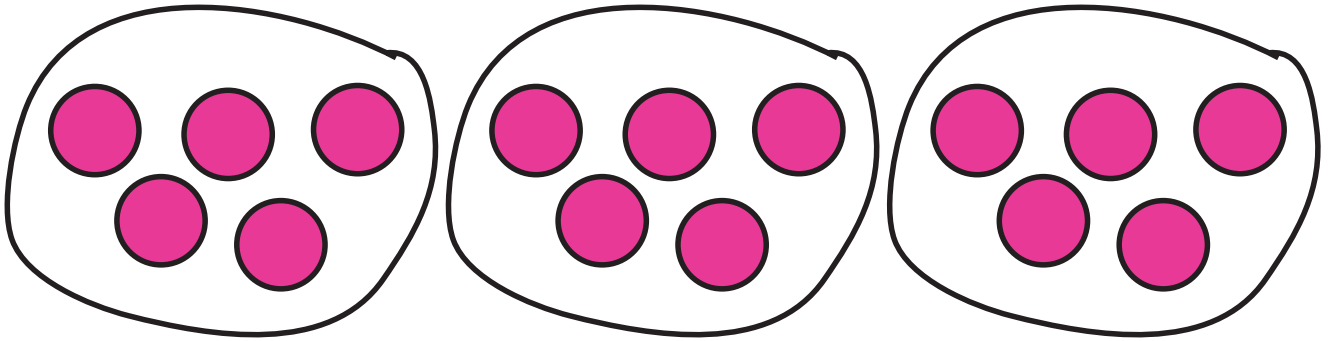
To teach the 3rd grade Common Core Standards, teachers must:

- appreciate that the commutative property is not obvious
- understand why the property is true for counting numbers
- recognize the importance of commutativity for developing fluency with single-digit multiplications.

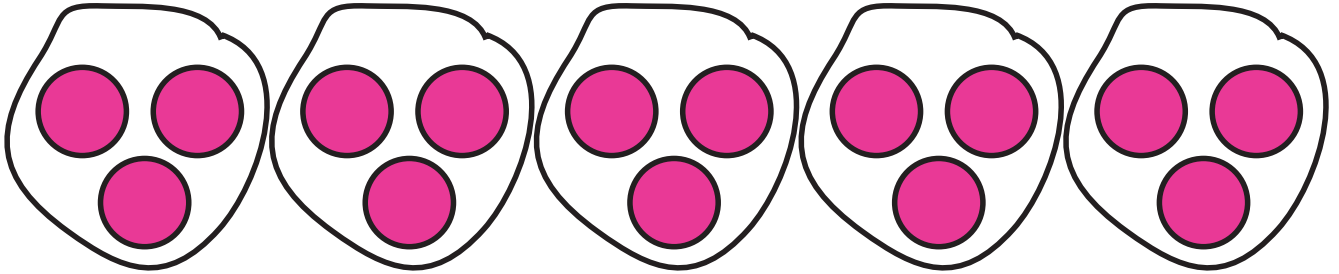
Grade 3: the commutative property of multiplication

A 3rd grade perspective on why the commutative property of multiplication is not obvious:

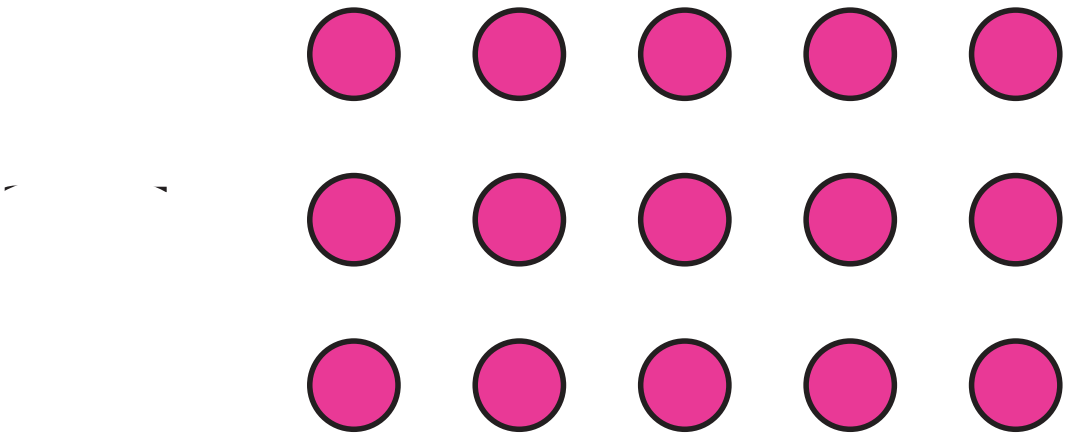
3×5



5×3



Grade 3: the commutative property of multiplication



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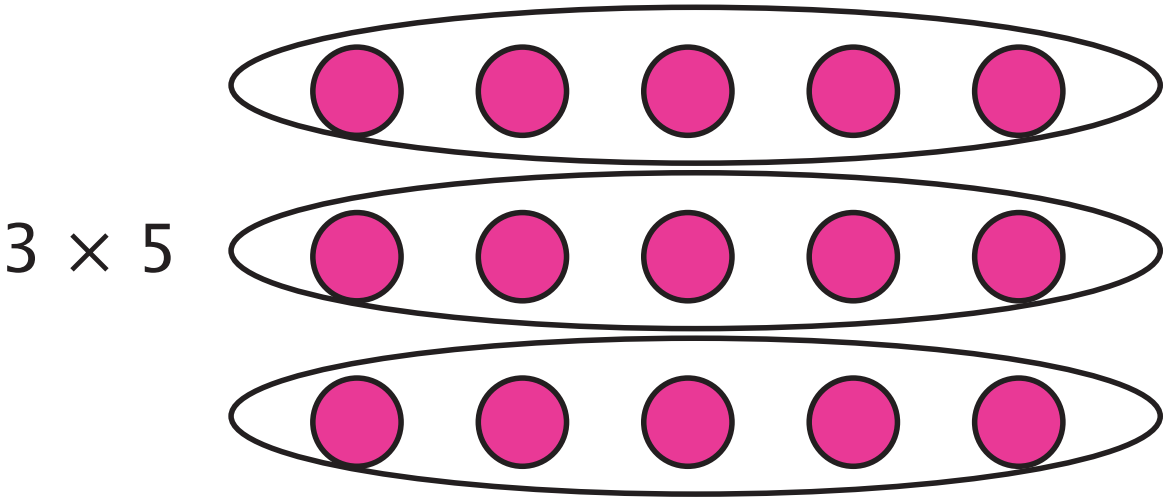
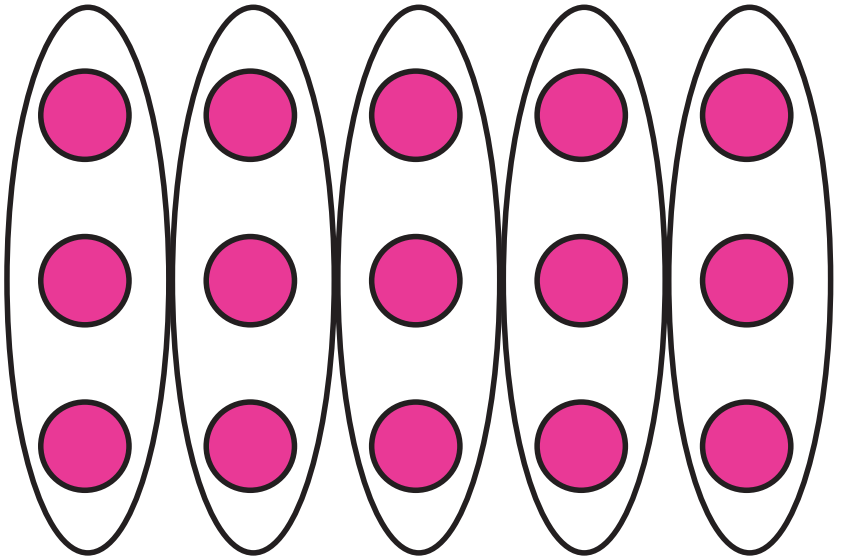


Grade 3: the commutative property of multiplication



5×3

Grade 3: Learning single-digit multiplication facts

Relationships among facts and patterns are important for scaffolding student learning for fluency.

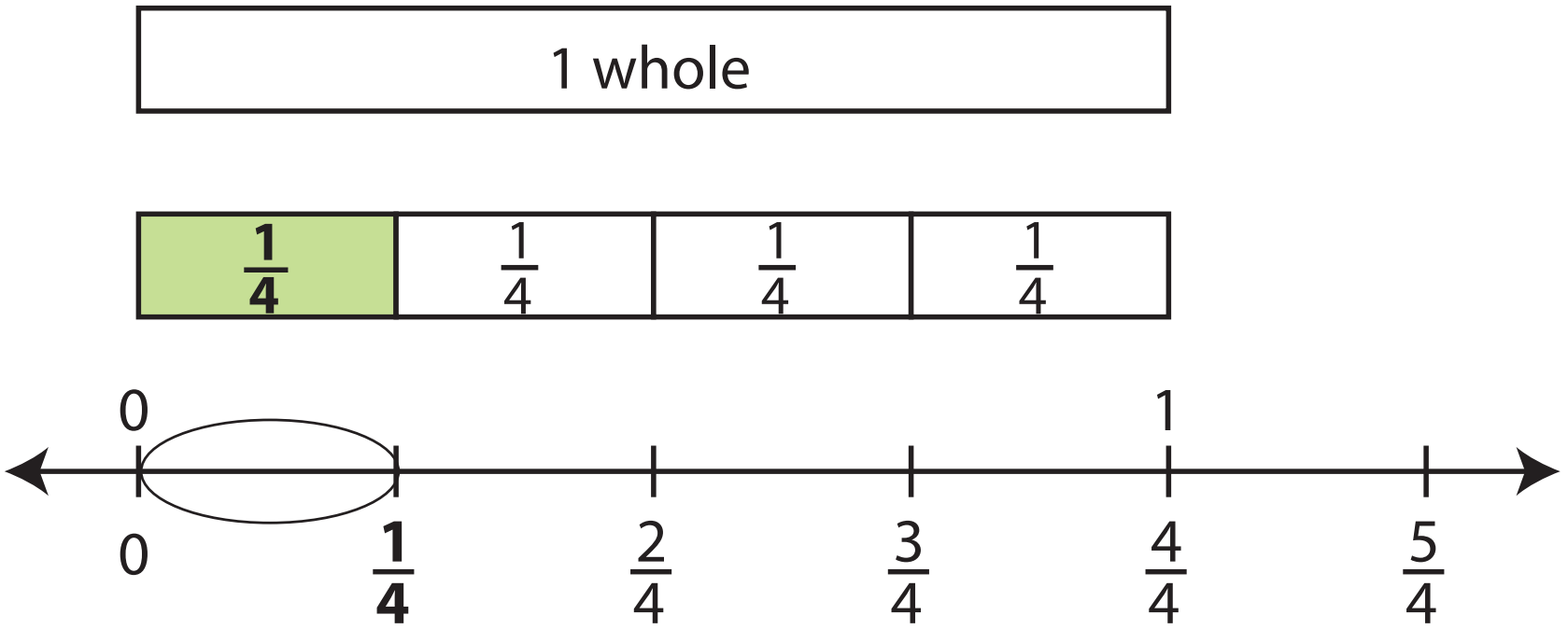
$6 \times 7 = 5 \times 7 + 1 \times 7$

$6 \times 7 = 2 \times (3 \times 7)$

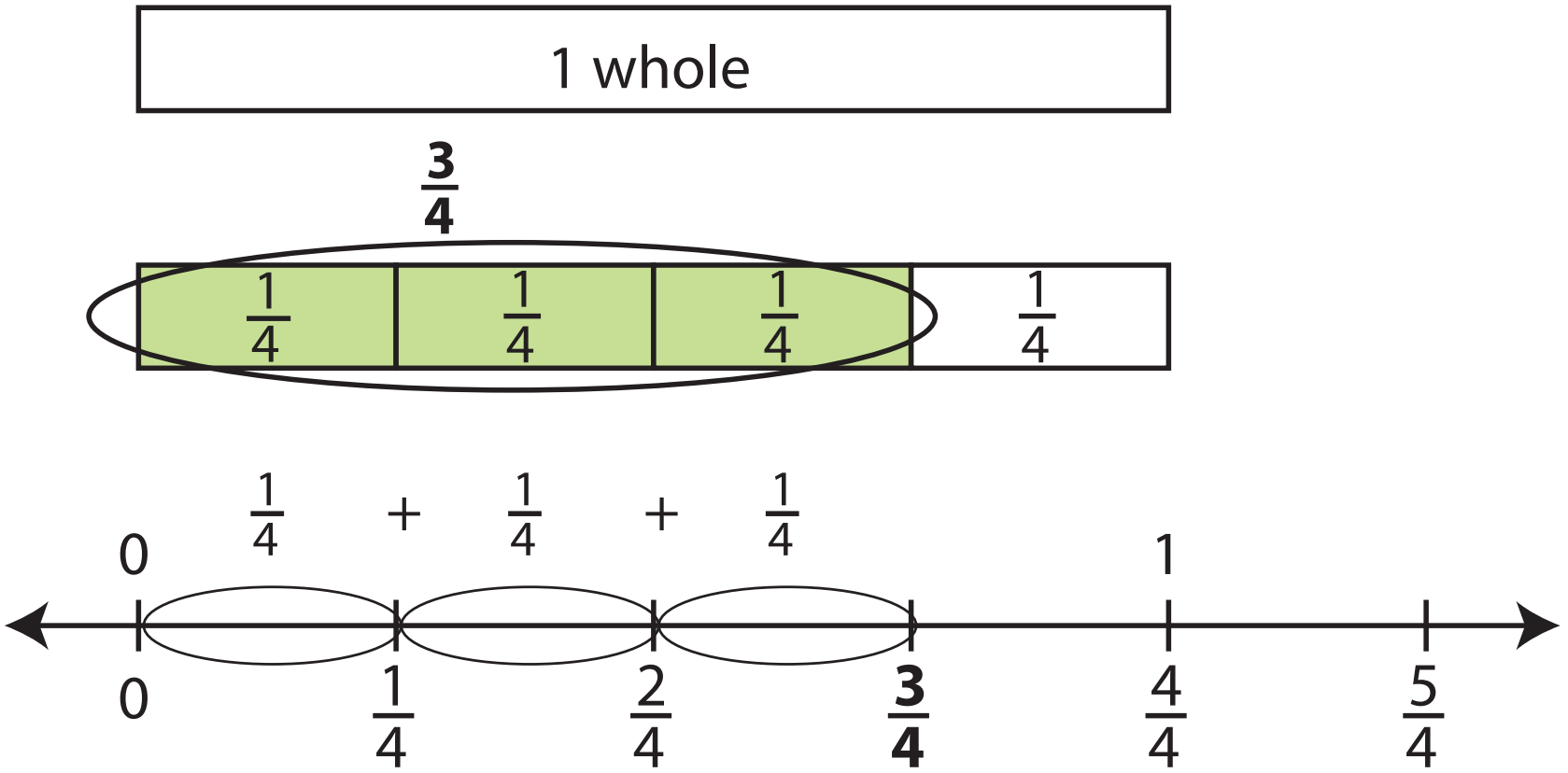
$6 \times 7 = 6 \times 5 + 6 \times 2$

Grade 3: Fractions

What do $\frac{1}{B}$ and $\frac{A}{B}$ mean?


1 whole
1/4
1/4
1/4
1/4
0
1
0
1/4
2/4
3/4
4/4
5/4

Grade 3: Fractions



Grade 4: Word problems using the four operations

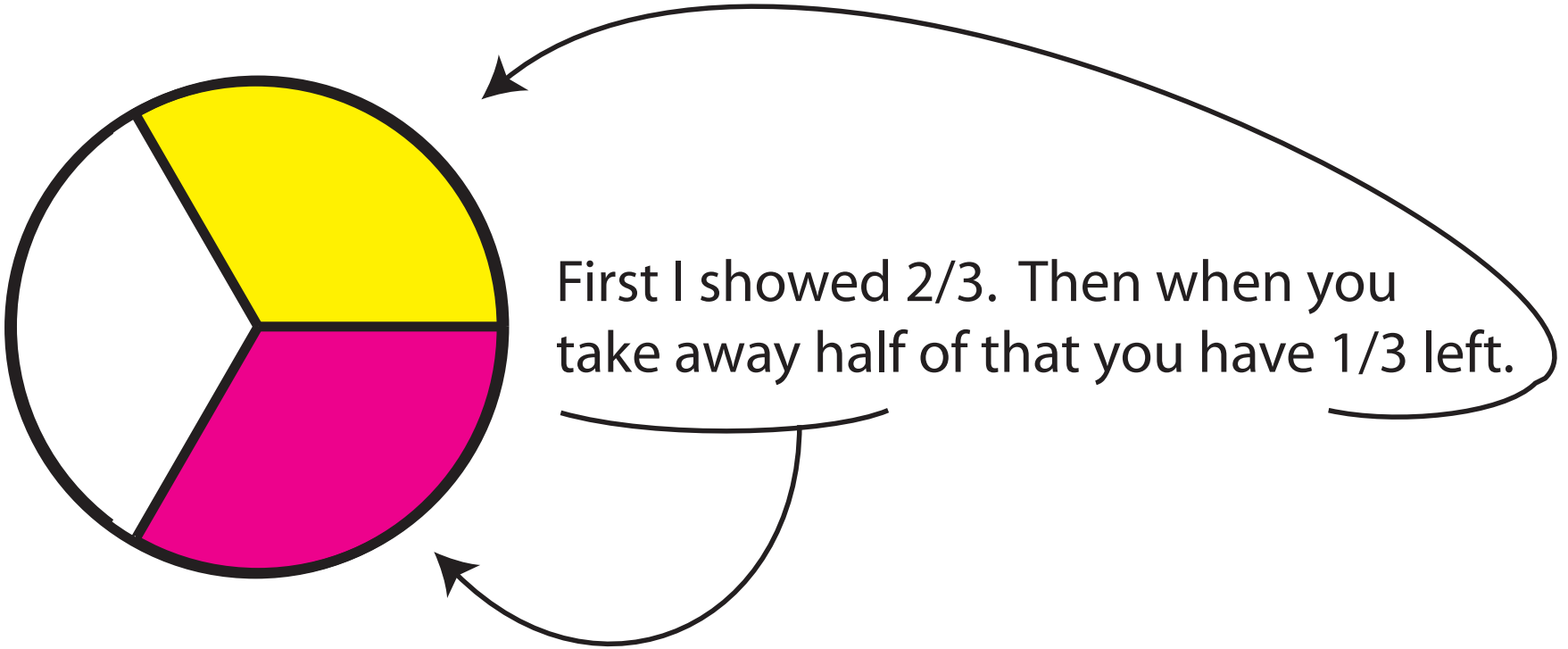
A grade 4 Common Core standard:

- Solve problems posed with both whole numbers and fractions. Understand that while quantities in a problem might be described with whole numbers, fractions, or decimals, the operations used to solve the problem depend on the relationships between the quantities regardless of which number representations are involved.

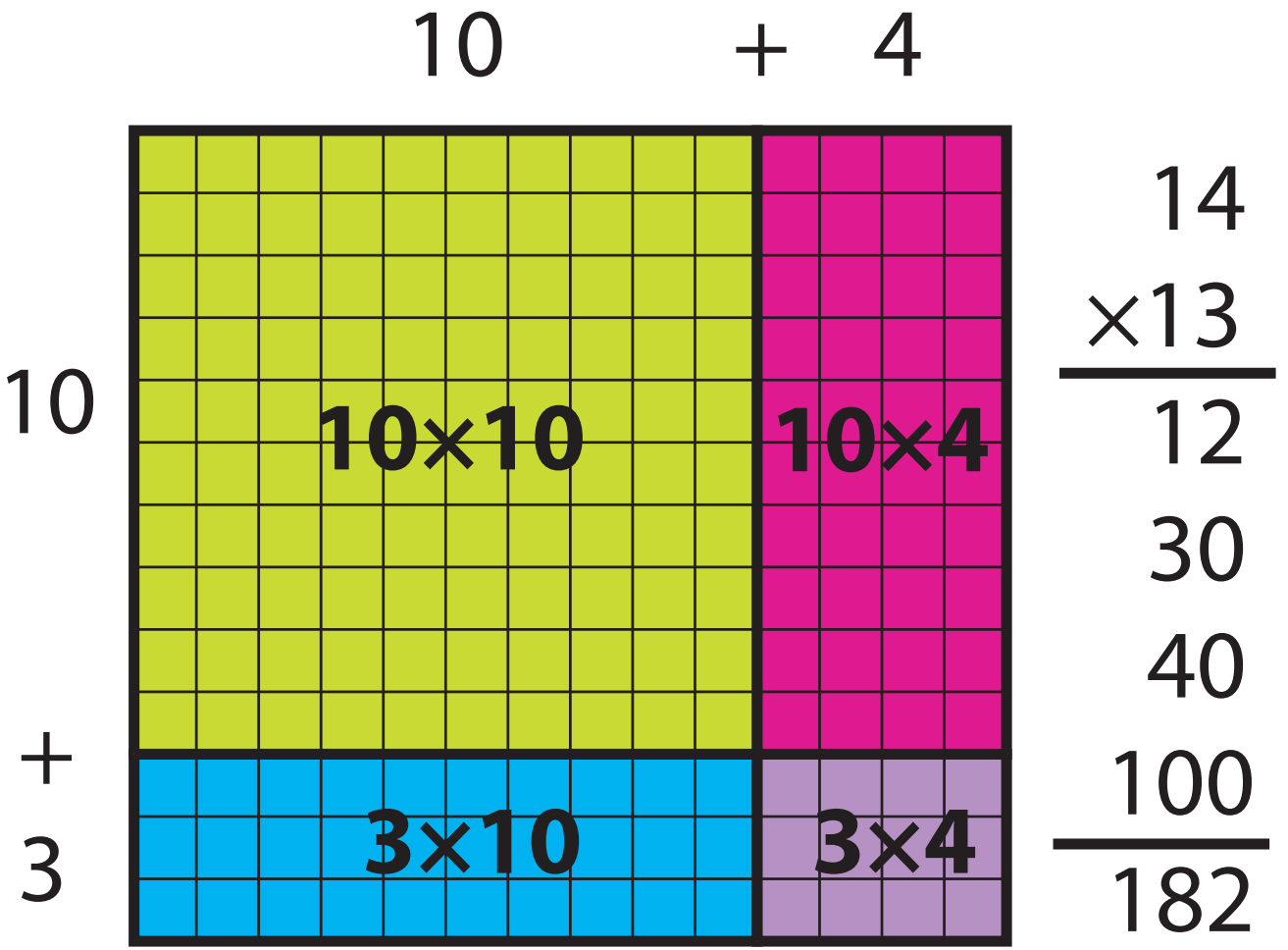
Fraction word problems

Is this a story problem for $\frac{2}{3} - \frac{1}{2}$?

There was $\frac{2}{3}$ of a cake left over. Claire ate $\frac{1}{2}$ of the cake that was left. Then how much cake was left?



Grade 4: Explaining a multiplication algorithm



Grades 3 and 4: Area

- Grade 3: Units of area
  A plane figure which can be covered without gaps or overlaps by n unit squares has an area of n square units.

  Common misconception: 2 square inches means the area of a 2 inch by 2 inch square.

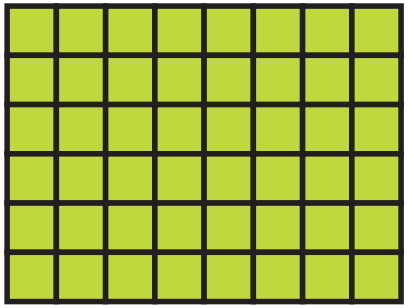
- Grade 4: Area formula for rectangles

Grades 3 & 4: Area of rectangles

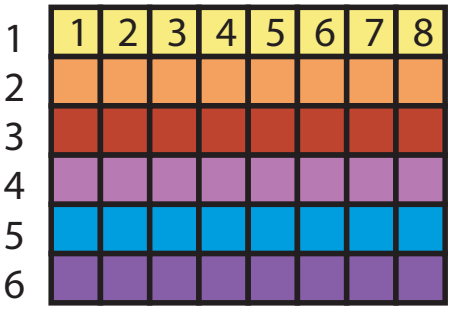
1 square unit

What is the area of this rectangle in square units?

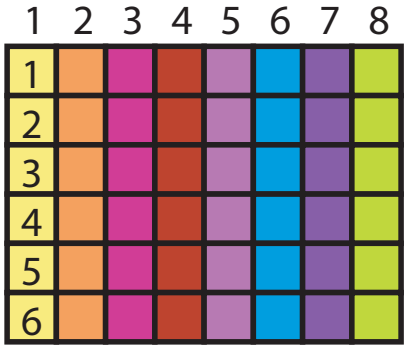
Cover the rectangle with squares. How many squares?



Is there a quicker way to find the area than counting all the squares one by one?



View as 6 groups of 8 squares.
$6 \times 8 = 48$ square units



View as 8 groups of 6 squares.
$8 \times 6 = 48$ square units

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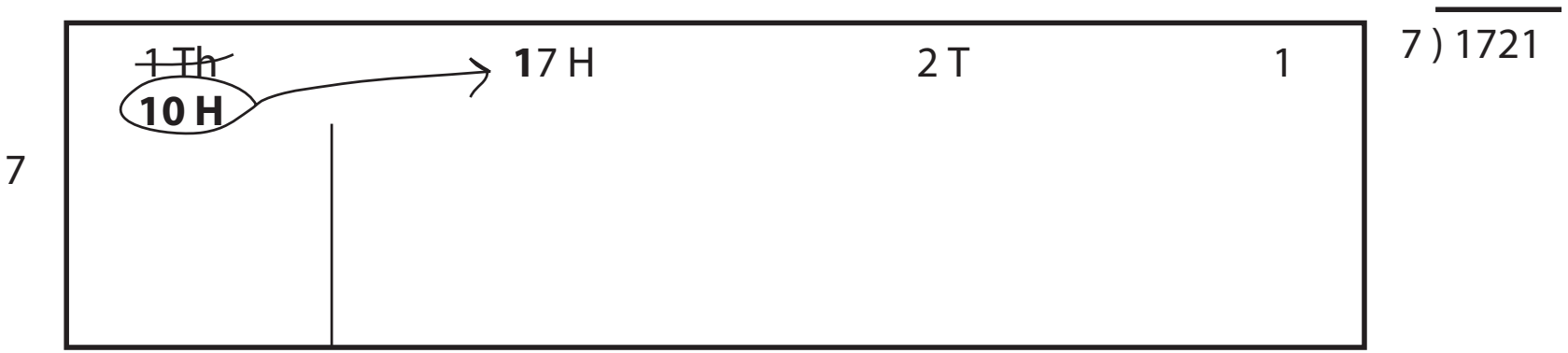
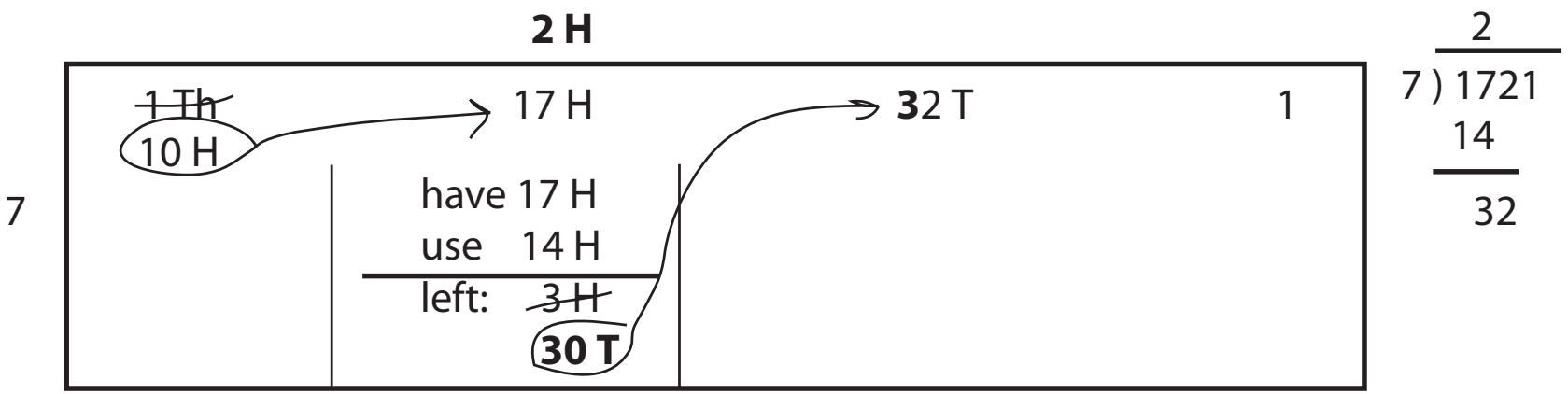
Grades 4 & 5: Explaining a division algorithm

STEP 1:



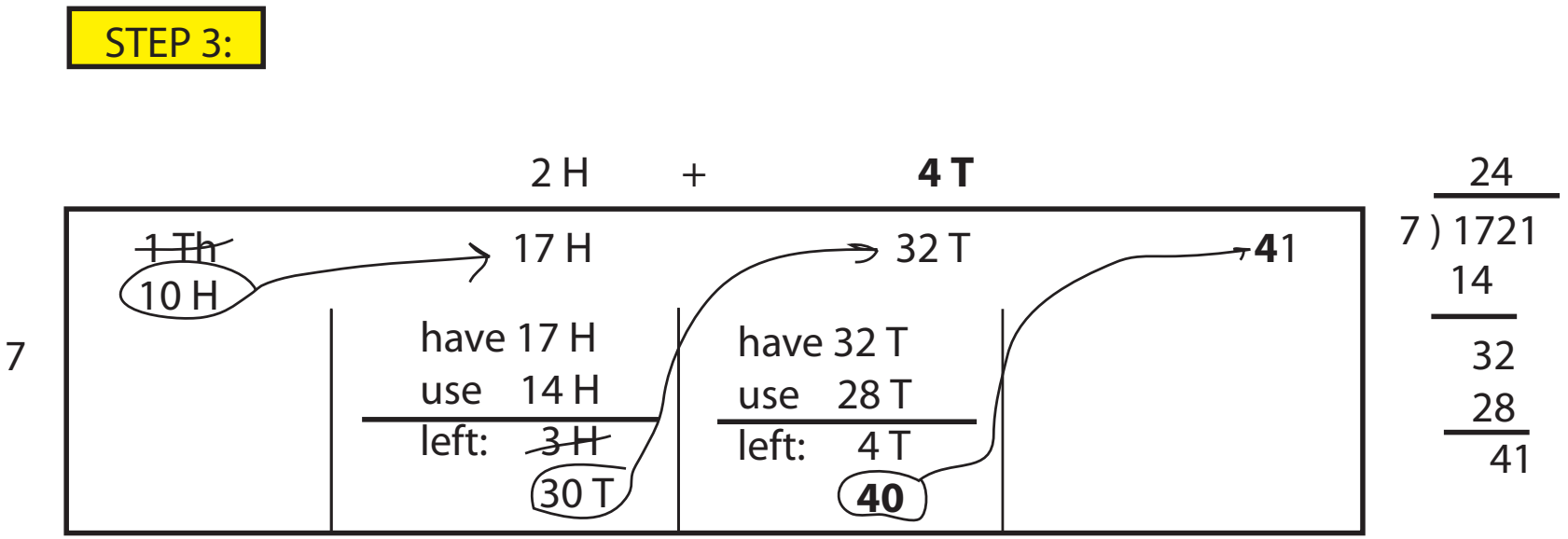
Grades 4 & 5: Explaining a division algorithm

STEP 2:



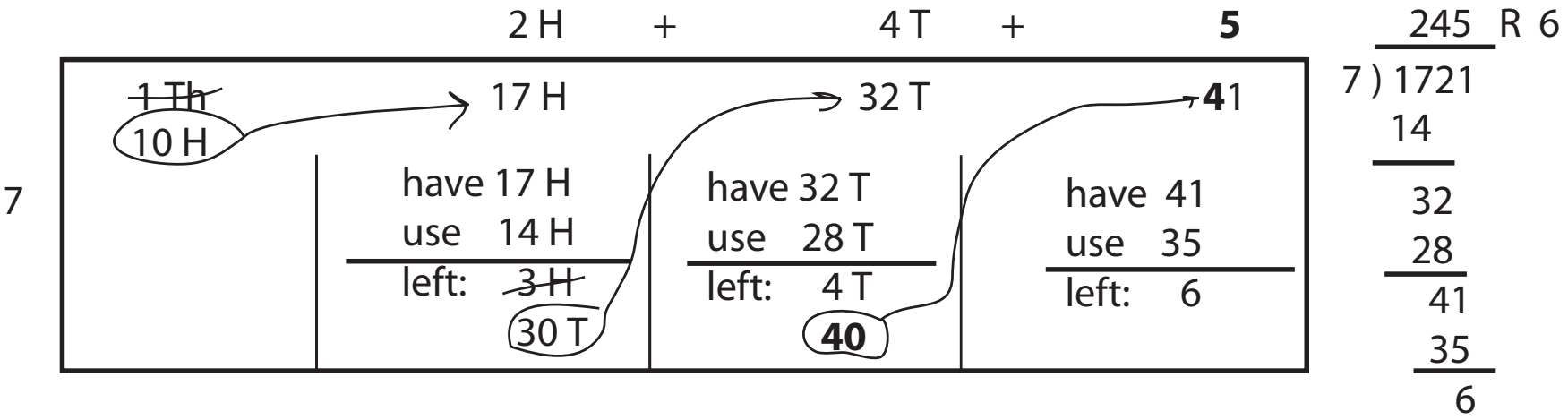
$$\begin{array}{r} 2 \\ \hline 7\,)\,1721 \\ 14 \\ \hline 32 \end{array}$$

Grades 4 & 5: Explaining a division algorithm



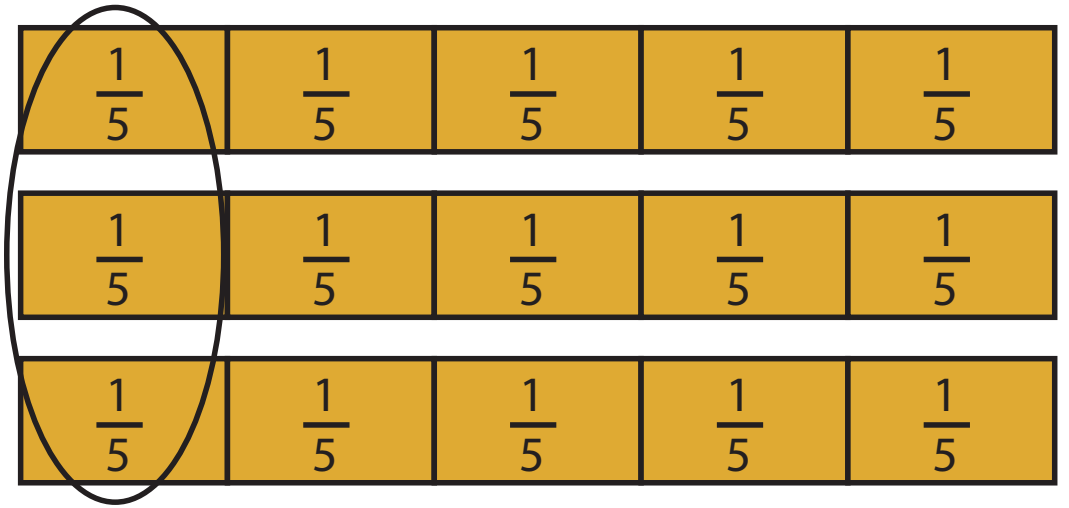
Grades 4 & 5: Explaining a division algorithm

STEP 4:



Grade 4: Explaining the connection between division and fractions

3 subs divided equally among 5 people



$$3 \div 5 = \frac{3}{5}$$

$$\frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \frac{3}{5}$$

1 person's share is 3/5 of a sub

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Explanations

- That procedures, formulas, and methods can be explained is usually unfamiliar to prospective teachers.
- Without suitable guidance, many explanations that prospective teachers will give are not based in logic but rather in mnemonics or analogies, i.e., they are not *mathematical* explanations, e.g., “take out” decimal places and “put them back in” when explaining decimal multiplication.
- Prospective teachers must learn explanations that can “travel into the classroom” – explaining why is not just part of more advanced math.

Development of a rationale

Teaching Mathematics in Seven Countries, Results from the TIMSS 1999 Video Study

Percentage of eighth-grade mathematics lessons in sub-sample that contained the development of a rationale, by country: 1999

Australia	25%
Switzerland	25%
Hong Kong	20%
Czech Republic	10%
Netherlands	10%
United States	0%

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Instructional conversations

Teachers must be prepared to have instructional conversations with their students about mathematical ideas.

The challenge for our teaching:

- For students to understand mathematical ideas requires that they engage with these ideas;
- our teaching must promote reasoning about, making sense of, and explaining ideas in addition to solving problems/exercises;
- students need time to explain to each other and to examine reasoning.

Grade 5: Equivalent fractions

How might a 5th grade teacher explain why

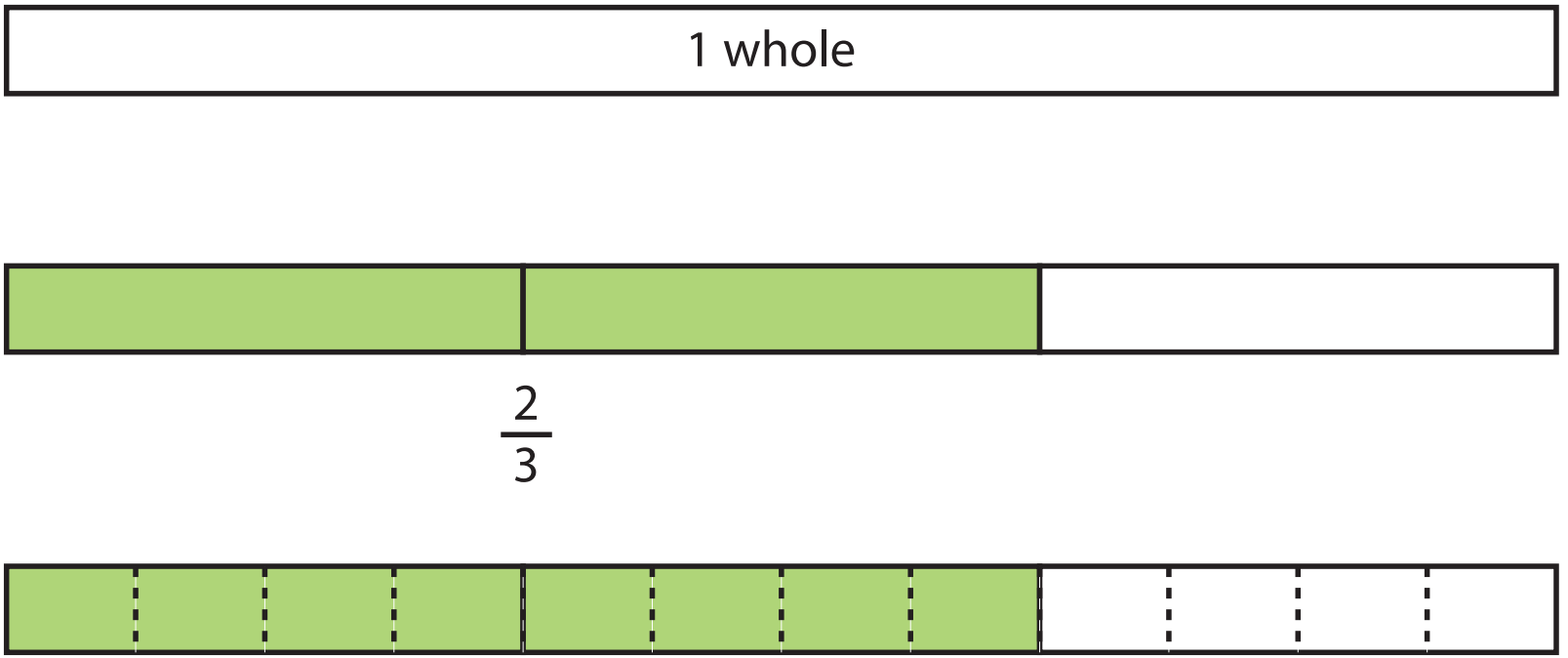
$$\frac{2}{3} = \frac{2 \cdot 4}{3 \cdot 4}?$$

Grade 5: Equivalent fractions

Students usually study fraction equivalence before they study fraction multiplication, so the teacher must know a different explanation from this one that uses multiplication by 1 in the form $\frac{4}{4}$:

$$\frac{2}{3} = \frac{2}{3} \cdot 1 = \frac{2}{3} \cdot \frac{4}{4} = \frac{2 \cdot 4}{3 \cdot 4}$$

Grade 5: Equivalent fractions



$$\frac{2}{3} = \frac{2\times4}{3\times4} = \frac{8}{12}$$

split each part into 4 parts
then there are 4 times as many shaded parts
and 4 times as many parts in all
but each part is only a fourth as big

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Grade 5: Fraction multiplication

Darrel has $\frac{1}{3}$ of a package of cheese left. He cuts off $\frac{1}{4}$ of it. What fraction of the package of cheese did he cut off?

"$\frac{1}{4}$ of $\frac{1}{3}$" is $\frac{1}{4} \times \frac{1}{3}$

just as

"4 of 3" is 4×3

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Grade 5: Fraction multiplication

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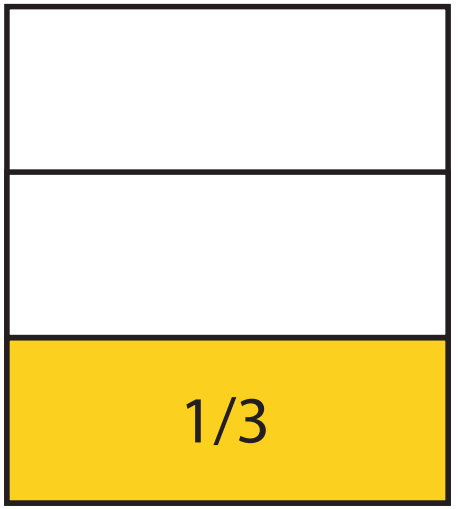
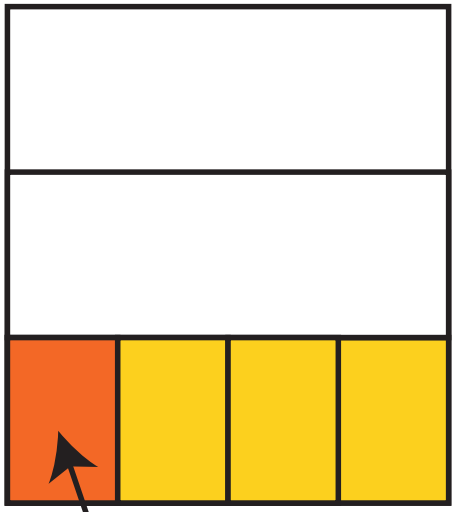
Grade 5: Fraction multiplication

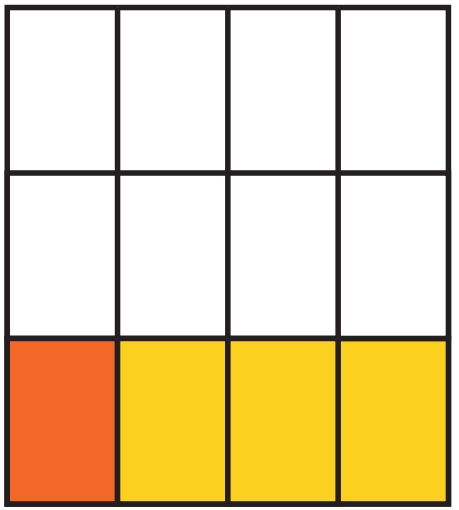




1/4 of 1/3

1/4 of 1/3 is 1/12

$1/4 \times 1/3 = 1/12$



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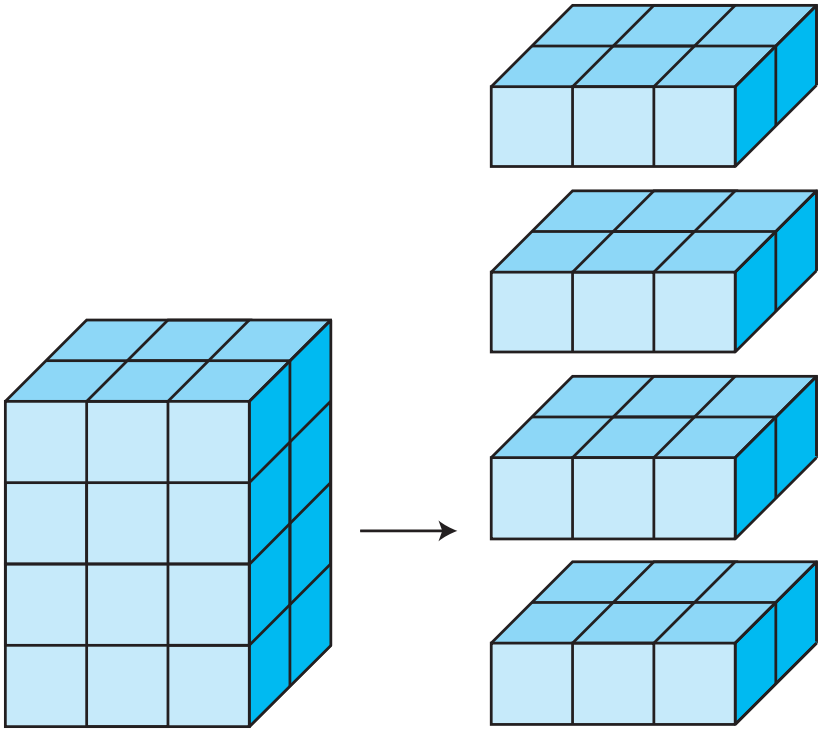
Grade 5: Volume

- A cube with side length 1 unit (a unit cube) is said to have “one cubic unit” of volume, and can be used to measure volume.
- The volume of a right rectangular prism with whole-unit side lengths can be found by packing it with unit cubes and using multiplication to count their number.

Grade 5: Explaining the volume formula for rectangular prisms



4 layers with
3×2 cubes in each layer

$4 \times (3 \times 2)$ cubes total

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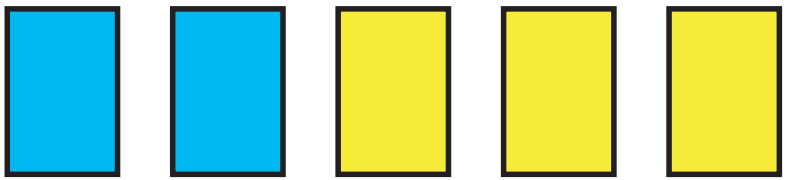
Grade 6: Ratios

Blue and yellow paint are mixed in a ratio of 2 to 3 to make green paint. How many pails of blue paint and how many pails of yellow paint will you need to make 30 pails of green paint?



# of batches	1	2	3	4	5	6	7
# pails blue paint	2	4	6	8	10	12	14
# pails yellow paint	3	6	9	12	15	18	21
# pails green paint produced	5	10	15	20	25	**30**	35

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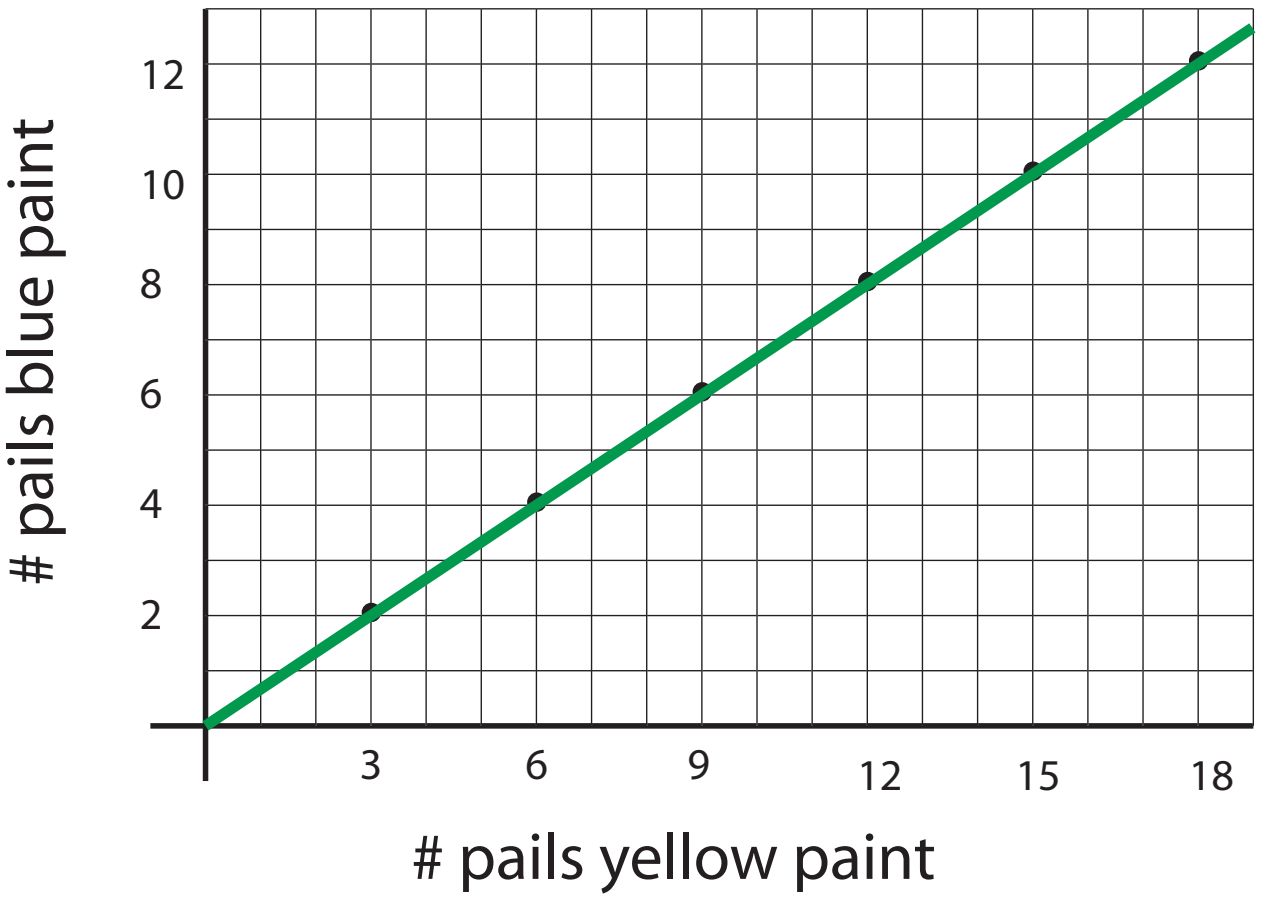
Graphing equivalent ratios



Reasoning about ratio tables

# of batches	1	2	3	4	5		
# pails blue paint	2	4	6	8	10		?
# pails yellow paint	3	6	9	12	15		?
# pails green paint produced	5	10	15	20	25		100

$$\frac{2}{5} = \frac{?}{100}$$

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Reasoning about ratio tables

# of batches	1	2	3	4	5		
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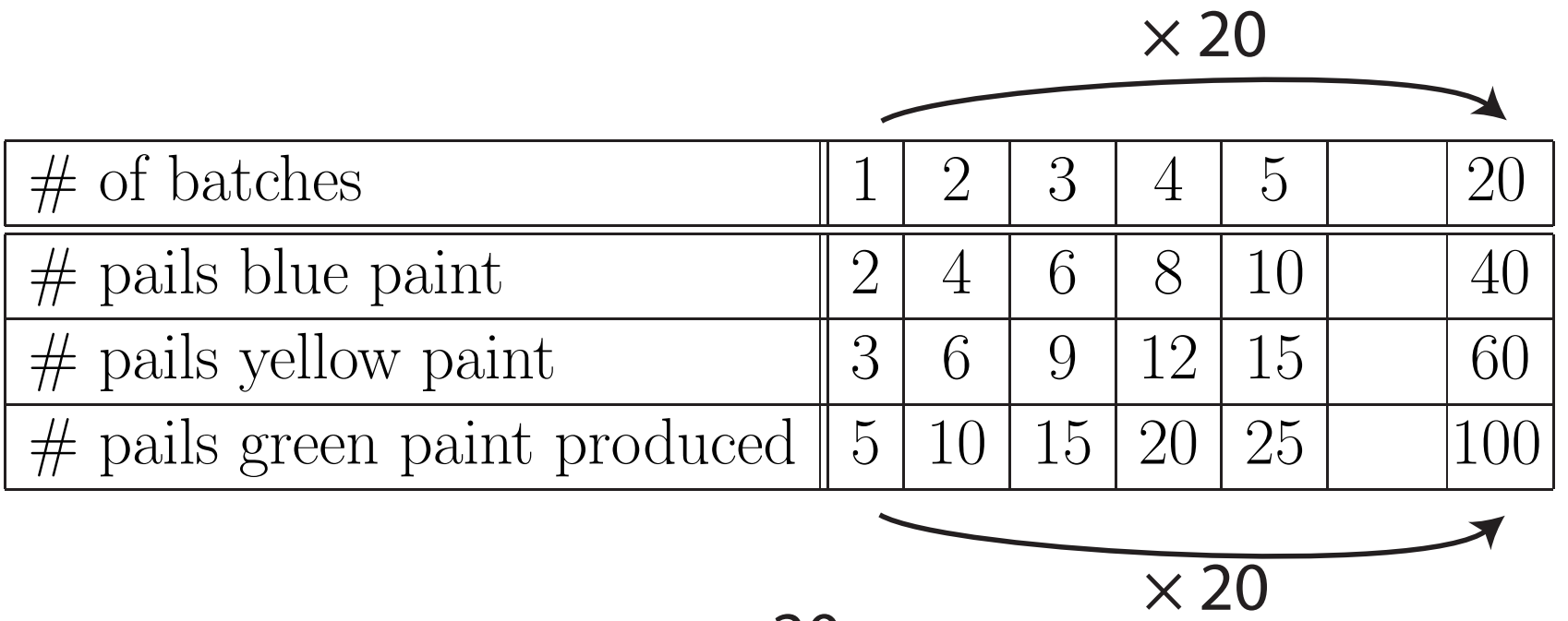
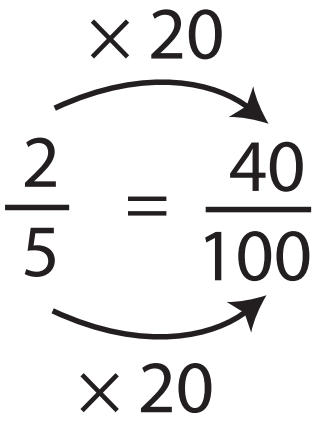
Reasoning about ratio tables





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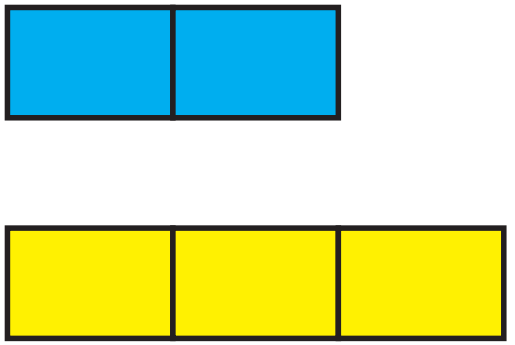

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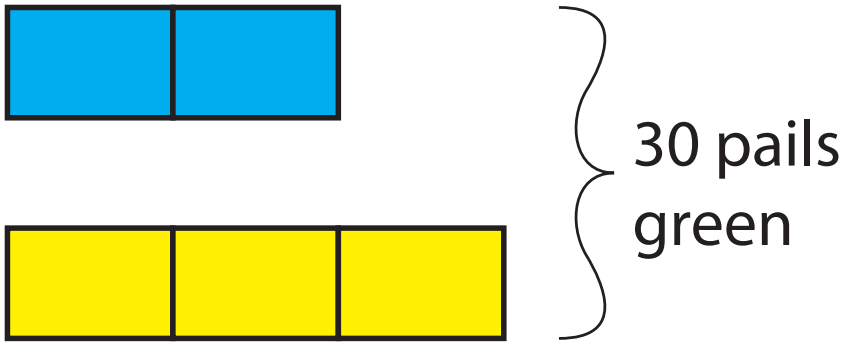

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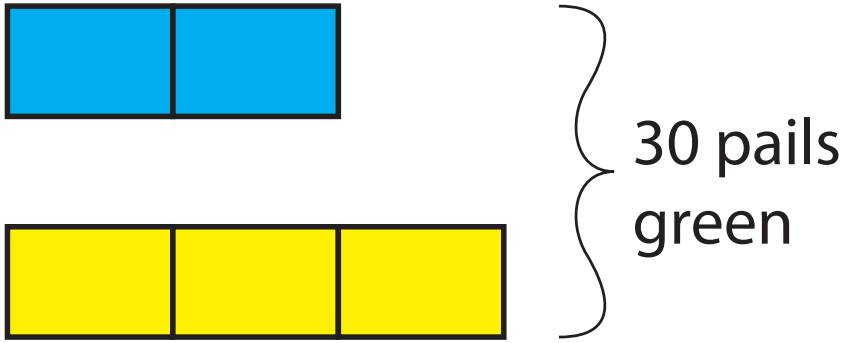

5 equal parts make 30 pails

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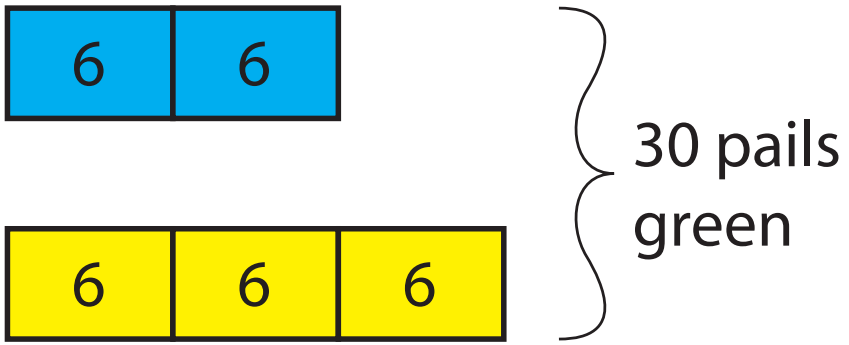

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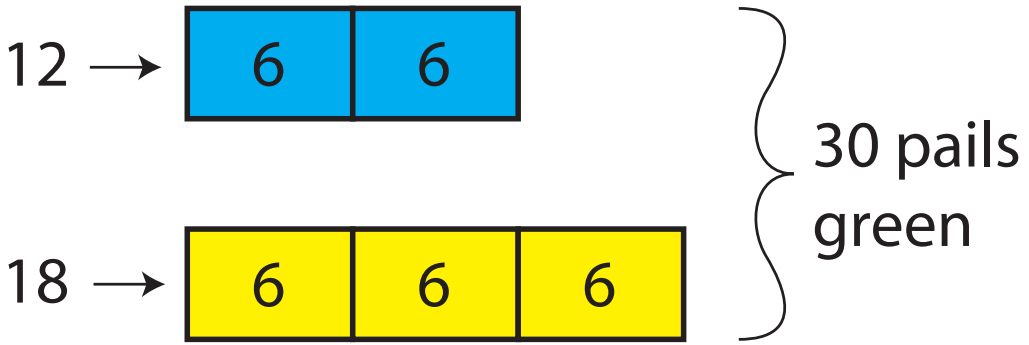

5 equal parts make 30 pails

An 8th grade TIMSS problem (1999)

A club has 86 members, and there are 14 more girls than boys. How many boys and how many girls are members of the club? Show your work.

% of 8th grade students solving correctly:

Singapore	72%
International average	33%
US	29%

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An 8th grade TIMSS problem (1999)

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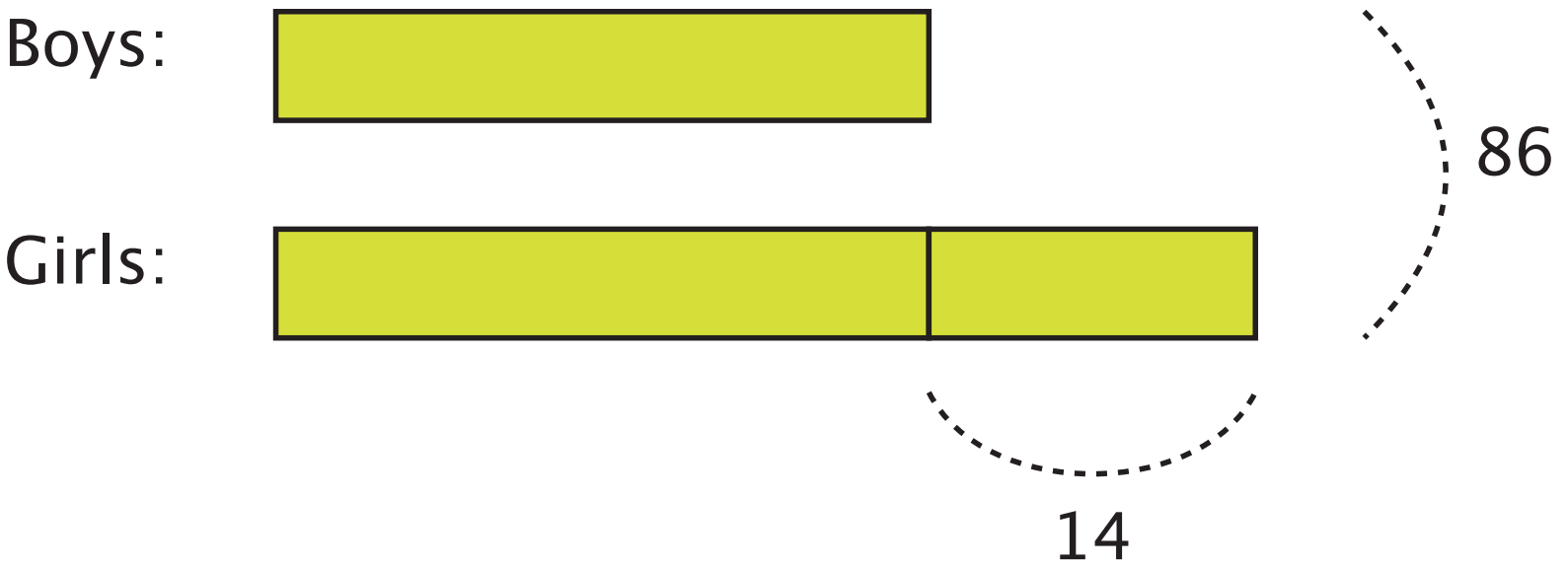

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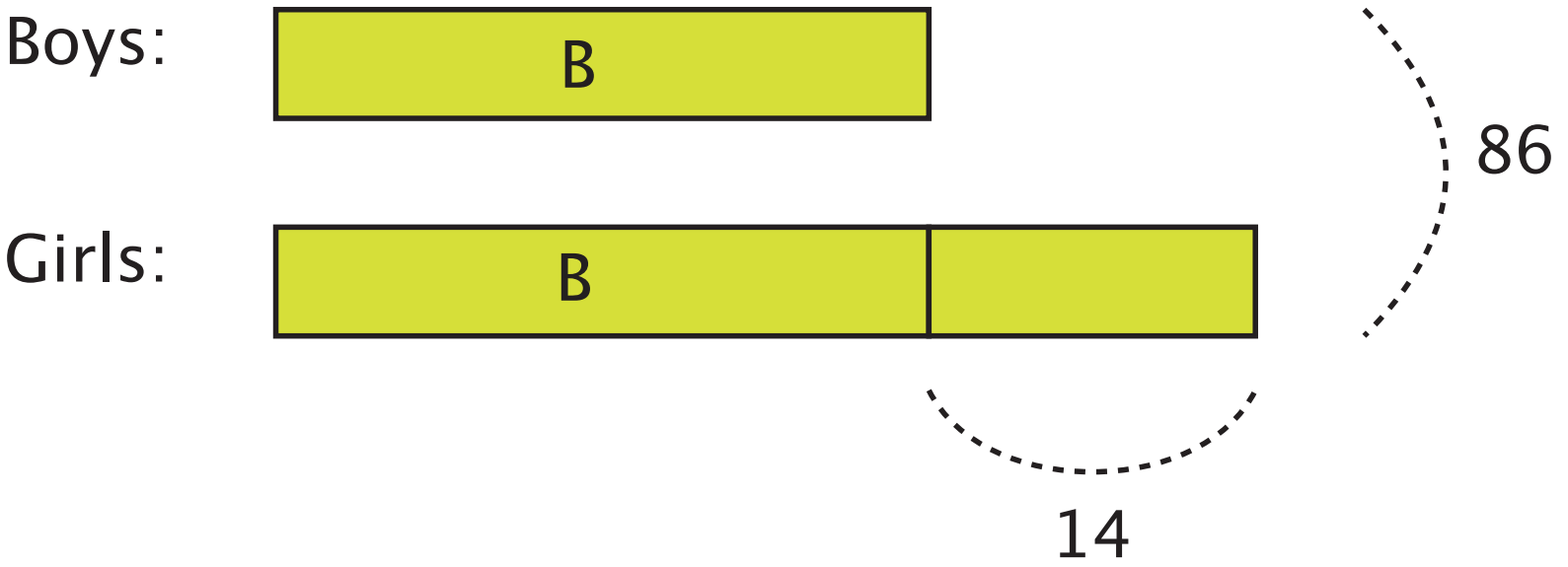


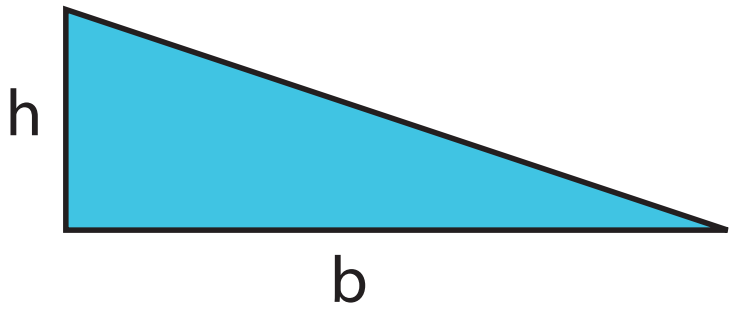
Let B be the number of boys. Then

$B + (B + 14) = 86$

$2B + 14 = 86 \ldots$

Grade 6: The area formula for triangles

One method:



Grade 6: The area formula for triangles

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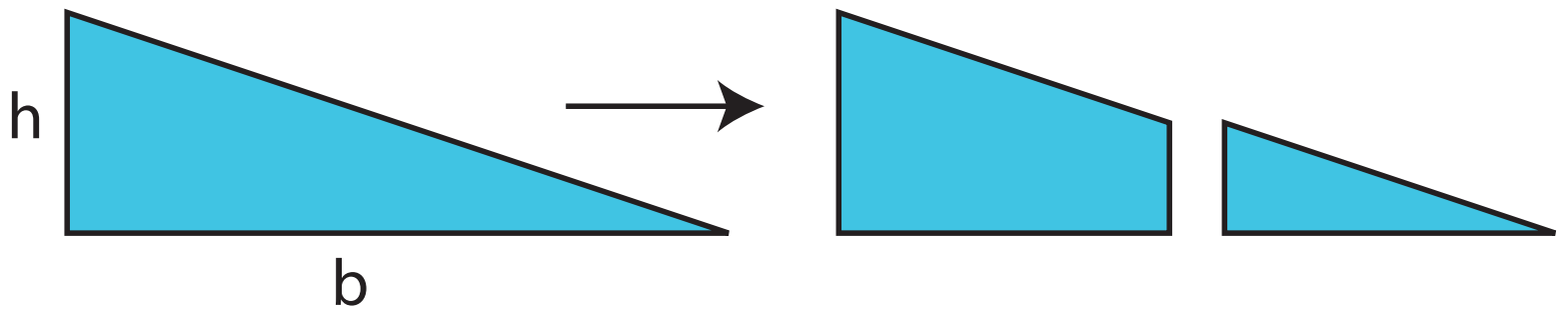

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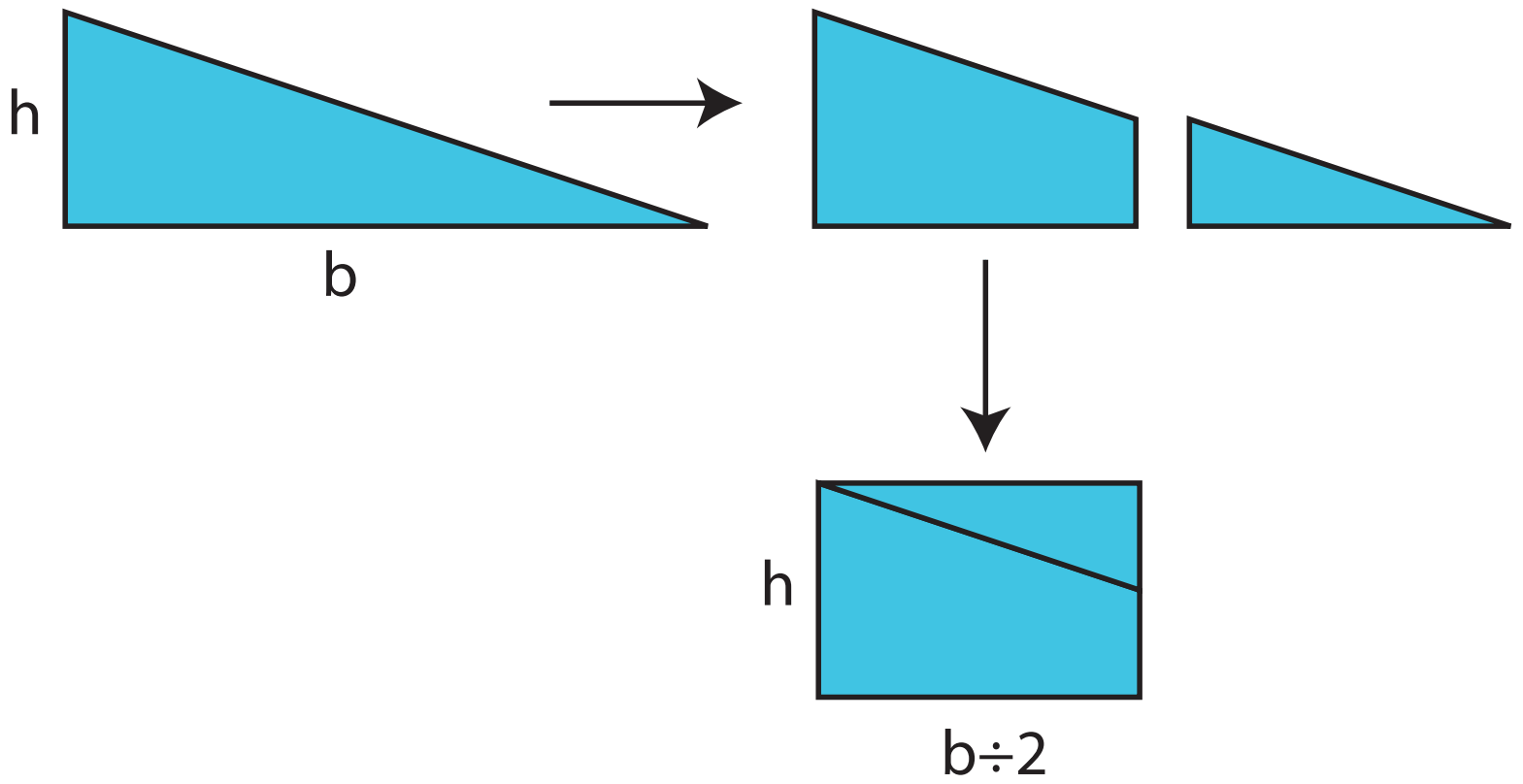
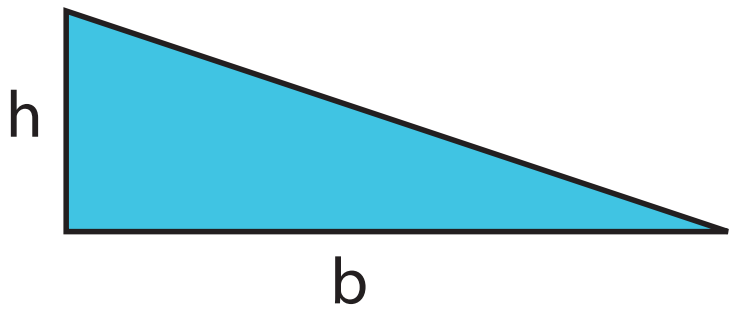


Grade 6: The area formula for triangles

Another method:



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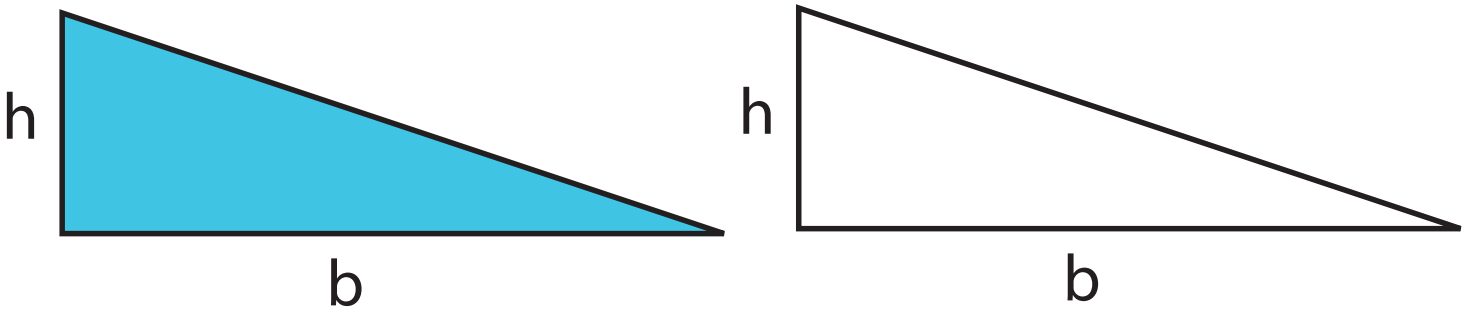

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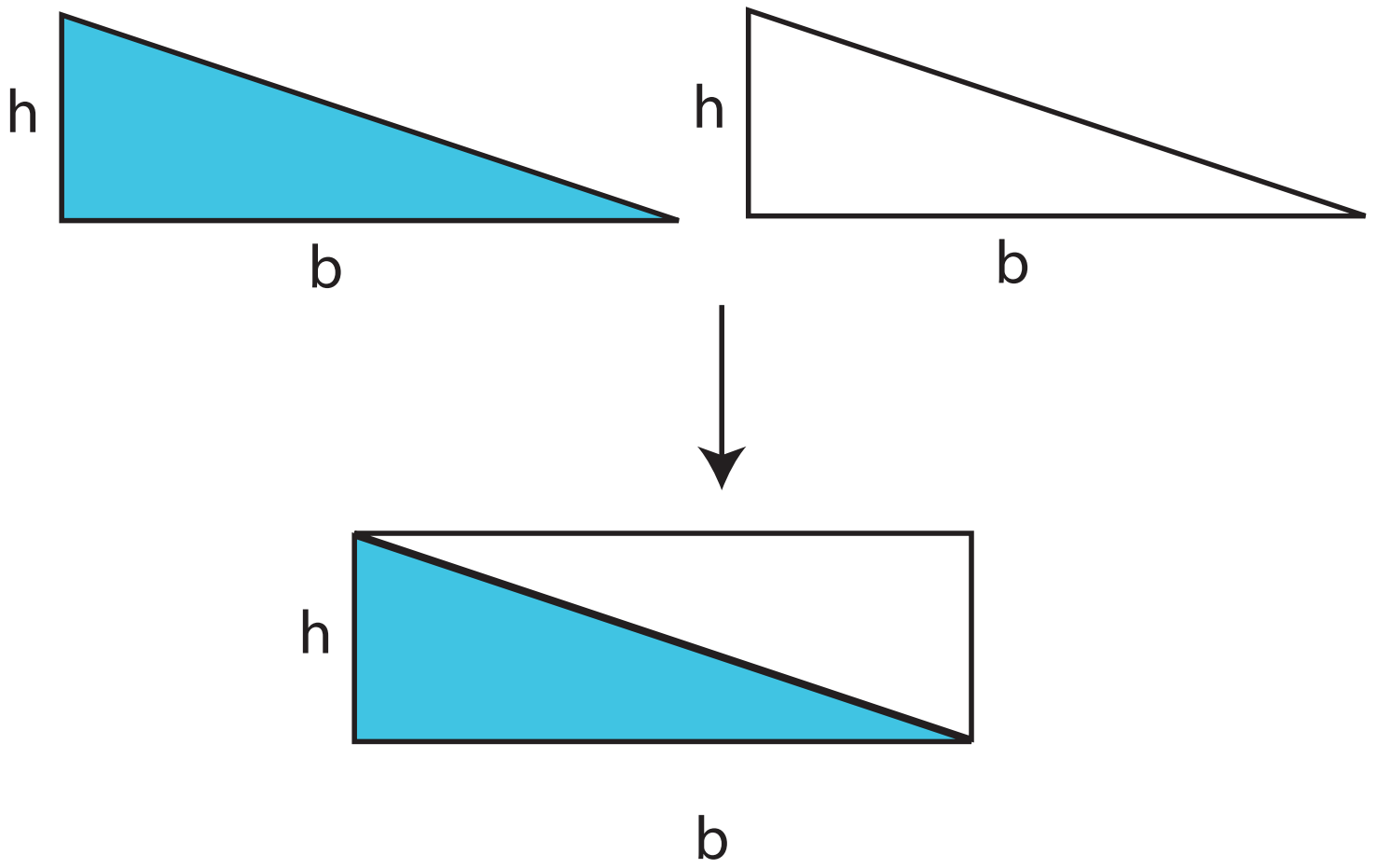
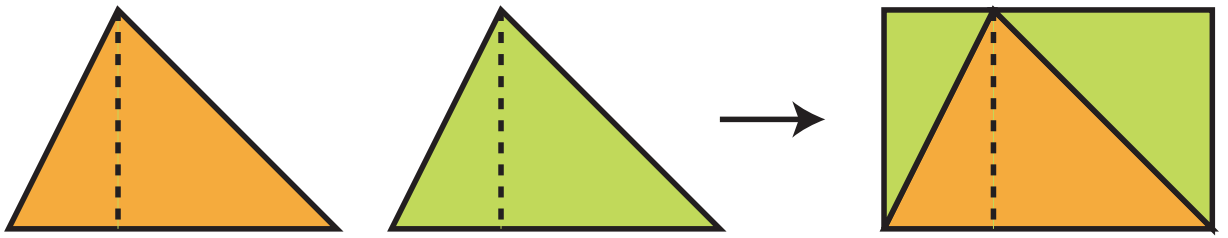


Similar reasoning used in geometry and in arithmetic

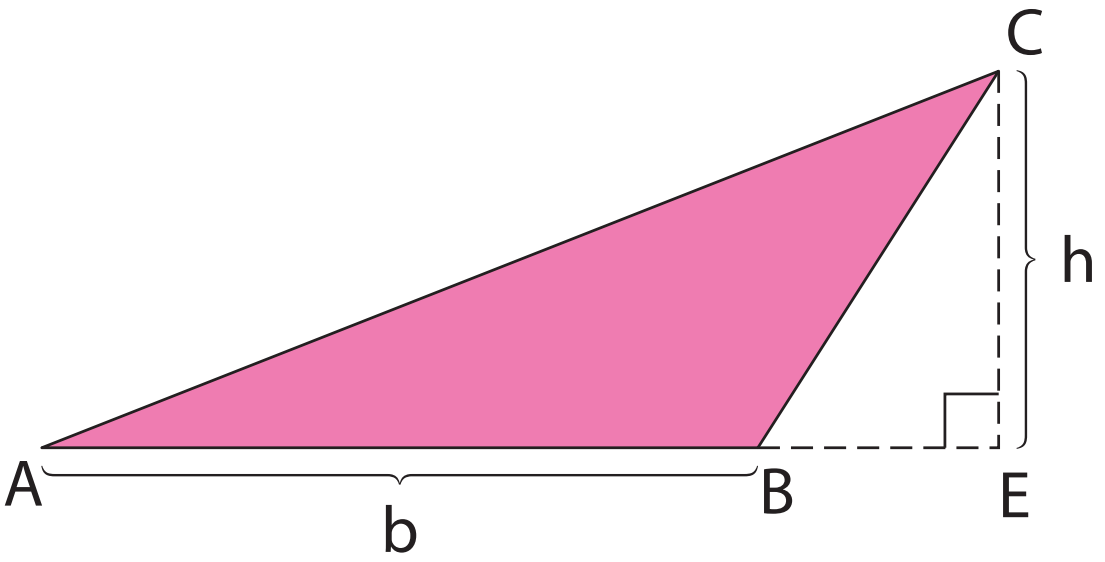


$$\updownarrow$$

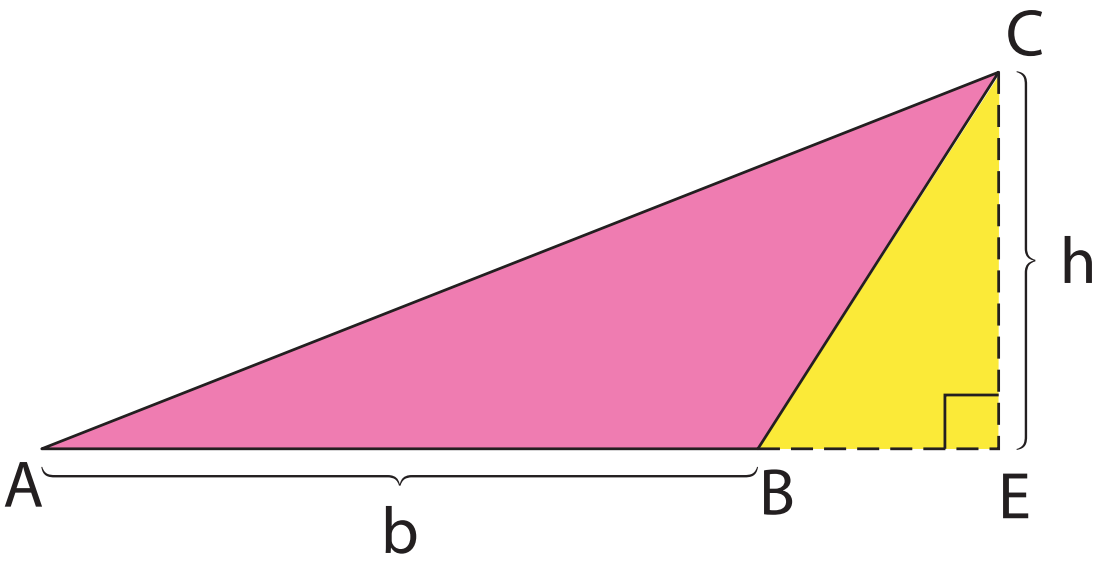
$$5 \times 86 = \frac{1}{2}(10 \times 86)$$

Grade 6: The area formula for triangles

Is the area formula still true for *this* base b and height h?



Grade 6: The area formula for triangles



view the oblique triangle as a “difference” of right triangles

Similar reasoning can be used in arithmetic

Problem: What is 45% of 120?

Student solution: Half of 120 is 60. Ten percent of 120 is 12, so 5% of 120 is half of that ten percent, which is 6. So the answer is 60 minus 6, which is 54.