Key Ideas of Elementary Mathematics

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Lesson Study Conference, May 2007

US curricula are unfocused

A Splintered Vision, 1997 report based on the TIMSS curriculum analysis.
US state math curriculum documents:

"The planned coverage included so many topics that we cannot find a single, or even a few, major topics at any grade that are the focus of these curricular intentions. These official documents, individually or as a composite, are unfocused. They express policies, goals, and intended content coverage in mathematics and the sciences with little emphasis on particular, strategic topics."

US instruction is unfocused

From **A Splintered Vision**:

"US eighth grade mathematics and science teachers typically teach far more topic areas than their counterparts in Germany and Japan."

"The five surveyed topic areas covered most extensively by US eighth grade mathematics teachers accounted for less than half of their year's instructional periods. In contrast, the five most extensively covered Japanese eighth grade topic areas accounted for almost 75 percent of their year's instructional periods."

Breaking the “mile-wide-inch-deep” habit

Every mathematical skill and concept

- has some useful application
- has some connection to other concepts and skills

So what mathematics should we focus on?

What focus?

Statistics and probability are increasingly important in science and in the modern workplace.

Should we therefore focus on statistics and probability in school mathematics?

Calculators are always at hand (cell phones), so paper and pencil calculations seem nearly obsolete.

Should school mathematics no longer cover paper and pencil calculations?

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Did reform lead to widening of curricula?

From **A Splintered Vision** (1997):

“Mathematics reform recommendations thus far seem to have affected curricula primarily by inclusion of additional topics, with a concomitant decrease in how focused these curricula are. This seems to result from our unwillingness to drop other topics when newer topics are added.”

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More students should be prepared to continue in math and science

From **Rising Above The Gathering Storm**: *Energizing and Employing America for a Brighter Economic Future* (2007):

"Having reviewed trends in the United States and abroad, the committee is deeply concerned that the scientific and technological building blocks critical to our economic leadership are eroding at a time when many other nations are gathering strength."

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From **Rising Above the Gathering Storm**, 2007

"Recommendation A: Increase America's talent pool by vastly improving K–12 science and mathematics education."

"Action A–3: Enlarge the pipeline of students who are prepared to enter college and graduate with a degree in science, engineering, or mathematics by increasing the number of students who pass AP and IB science and mathematics courses."

"Action C–1: Increase the number and proportion of US citizens who earn bachelor's degrees in the physical sciences, the life sciences, engineering, and mathematics . . . "

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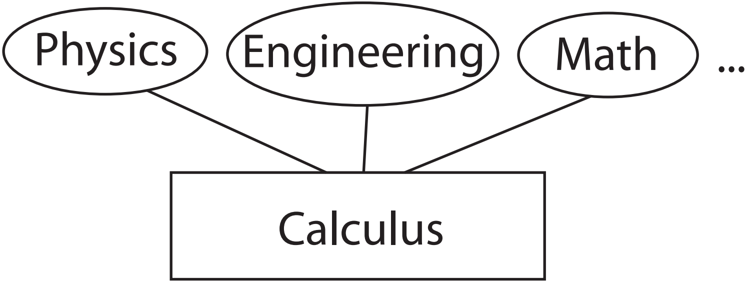
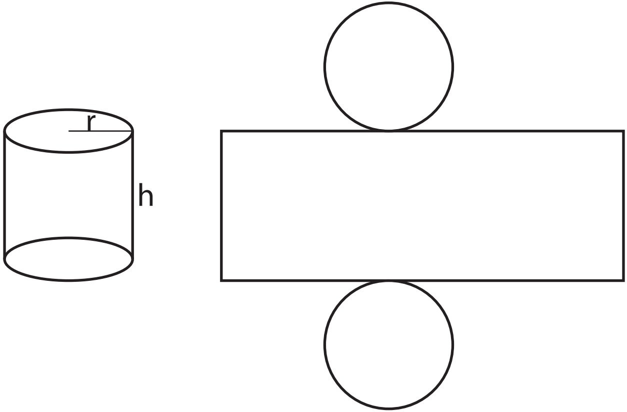
College math and beyond: calculus is foundational

Calculus provides a theoretical framework for many parts of physics and engineering, it is a foundation for additional mathematics and statistics, and it is used in chemistry and biology.



Among all cylindrical cans that hold a fixed amount, which can requires the least amount of metal to make?



Typical calculus tasks

- Find a formula for a function based on information about a situation, often making use of the Pythagorean theorem, similar triangles, area formulas, and volume formulas
- Find equations relating variables in a situation
- Given an equation in several variables, solve for one variable in terms of the others
- Take derivatives of functions
- Set the derivative of a function equal to 0 and solve the resulting equation

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What is needed to succeed in calculus?

To be prepared for success in calculus, students must:

- be proficient with arithmetic
- be proficient with algebra:
  - algebraic manipulations
  - set up and solve equations
  - work with functions, especially linear and quadratic functions
- be proficient with geometry/measurement:
  - area and volume
  - Pythagorean theorem
  - similarity
  - trigonometry
- solve multi-step problems
  - be able to "take apart, analyze, put back together"
  - visualize and draw simple pictures to capture a situation for mathematical analysis
- throughout: use logical reasoning and understand *why* methods work

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NCTM Curriculum Focal Points
for Prekindergarten through Grade 8 Mathematics

National Council of Teachers of Mathematics (NCTM), 2006

Focal Points:

- the most significant mathematical concepts and skills at each grade level
- the vast majority of instructional time should be spent on the focal points
- 3 at each grade level

Connections to the Focal Points:

- related content, including contexts for the focal points
- continuing development of topics in focal points of previous grades

Some aspects of the Focal Points

- Solid preparation for future mathematics, in particular, for algebra
- Numbers & operations and geometric measurement are emphasized; algebra gradually receives more emphasis in the later grades
- The need for quick recall of basic facts is stated explicitly
- Algorithms of arithmetic:
  - develop them
  - understand them in terms of place value and properties of operations
  - develop fluency
  - use them to solve problems
- Overall: a blend of skills and understanding

Mathematical Proficiency
(National Research Council, 2001)

Adding It Up

Mathematical Proficiency:

- conceptual understanding
- procedural fluency
- strategic competence
- adaptive reasoning
- productive disposition

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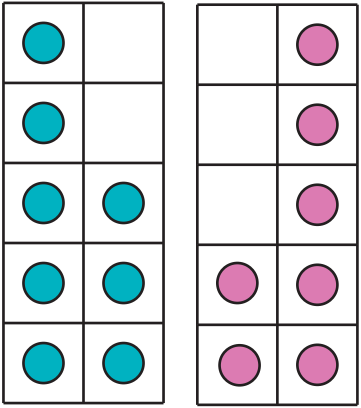
Developing understandings of addition and subtraction and strategies for basic addition facts and related subtraction facts

Children develop strategies for adding and subtracting whole numbers . . . They use a variety of models . . . to develop an understanding of the meanings of addition and subtraction and strategies to solve such arithmetic problems.
They use properties of addition (commutativity and associativity) to add whole numbers, and they create and use increasingly sophisticated strategies based on these properties (e.g., “making tens”) to solve addition and subtraction problems involving basic facts.

A basic addition fact

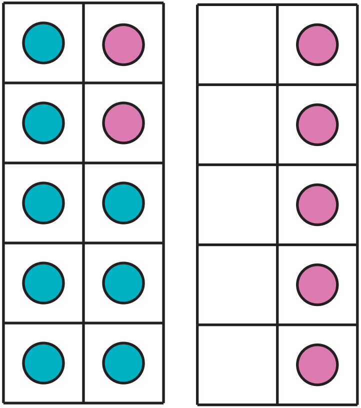
via “Make a Ten” with the associative property

$$8 + 7$$



A basic addition fact
via "Make a Ten" with the associative property

$$8 + 7 = 8 + (2 + 5) = (8 + 2) + 5 = 15$$



Understanding basic facts

1st graders are expected to understand basic addition and subtraction facts by using

- what addition and subtraction mean
- decomposing and composing (associative and commutative properties)
- place value — making a ten and some ones

A 4th grade Focal Point

Developing quick recall of multiplication facts and related division facts and fluency with whole number multiplication

Students use understandings of multiplication to develop quick recall of the basic multiplication facts and related division facts. They apply their understanding of models for multiplication (i.e., equal-sized groups, arrays, area models, equal intervals on the number line), place value, and properties of operations (in particular, the distributive property) as they develop, discuss, and use efficient, accurate, and generalizable methods to multiply multidigit whole numbers. . . .
. . . They develop fluency with efficient procedures, including the standard algorithm, for multiplying whole numbers, understand why the procedures work (on the basis of place value and properties of operations), and use them to solve problems.

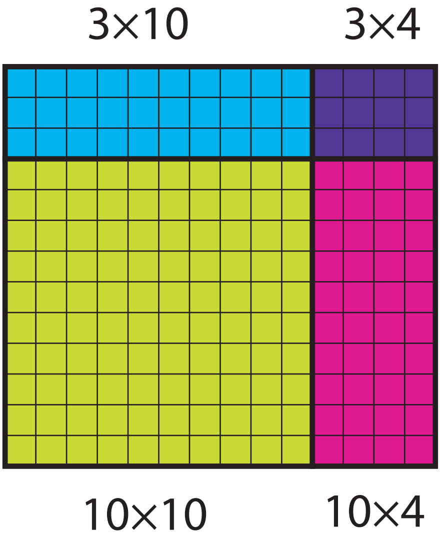
Understanding *why* procedures work

4th graders are expected to understand why multiplication procedures work based on

- what multiplication means
- place value
- the distributive property (decomposing and composing)

The Multiplication Algorithm

```
  14
 ×13
----
  12
  30
  40
 100
----
 182
```



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Assertions should have reasons

From NCTM **Principles and Standards for School Mathematics** (PSSM), 2000:

"From children's earliest experiences with mathematics, it is important to help them understand that assertions should always have reasons. Questions such as "Why do you think it is true?" and "Does anyone think the answer is different, and why do you think so?" help students see that statements need to be supported or refuted by evidence." (chapter 3)

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Reasoning is essential to understanding math

From PSSM:

“Being able to reason is essential to understanding mathematics. By developing ideas, exploring phenomena, justifying results, and using mathematical conjectures in all content areas and—with different expectations of sophistication—at all grade levels, students should see and expect that mathematics makes sense.”
(chapter 3)

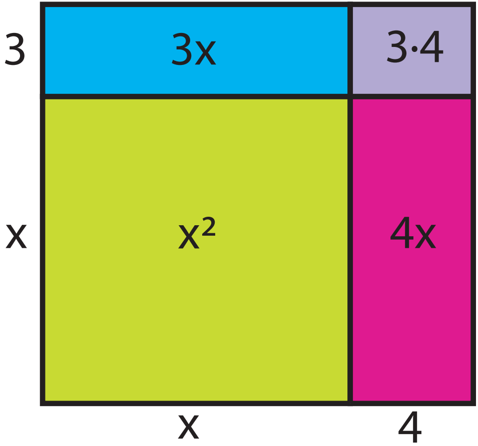
Development of a rationale

Teaching Mathematics in Seven Countries: *Results From the TIMSS 1999 Video Study*

Percentage of 8th grade mathematics lessons in sub-sample that contained the **development of a rationale**	
Australia	25%
Switzerland	25%
Hong Kong SAR	20%
Czech Republic	10%
Netherlands	10%
United States	**0%**

Understanding *why* the multiplication algorithm works connects arithmetic to algebra

$$(x + 3)(x+4) = x^2 + 3x + 4x + 3\cdot4$$



Reasoning about area and volume in the Focal Points

The Focal Points ask students not just to know and be able to use area and volume formulas but also to understand where these formulas come from by decomposing and composing:

- 3rd grade: “Students investigate, describe, and reason about decomposing, combining, and transforming polygons to make other polygons.”
- 4th grade: “Students connect area measure to the area model that they have used to represent multiplication, and they use this connection to justify the formula for the area of a rectangle.”
- 5th grade: “. . . [students] find and justify relationships among the formulas for the areas of different polygons.”
- 7th grade: “As students decompose prisms and cylinders by slicing them, they develop and understand formulas for their volumes ($\text{Volume} = \text{Area of base} \times \text{Height}$).”

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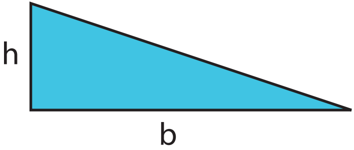
Explaining why the area formula for triangles is valid
by decomposing and composing to create a related rectangle

One method:



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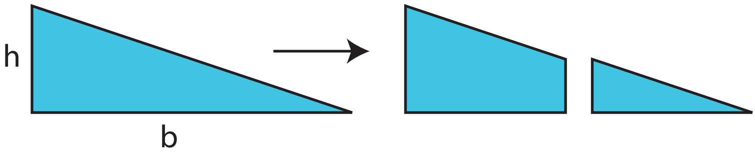

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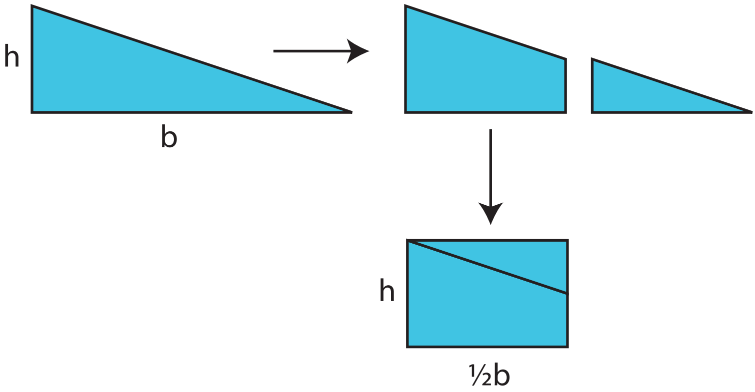

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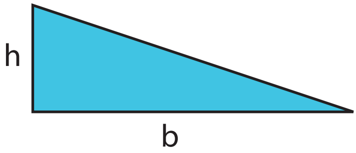

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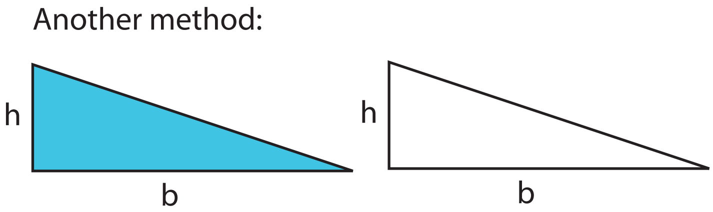

Explaining why the area formula for triangles is valid

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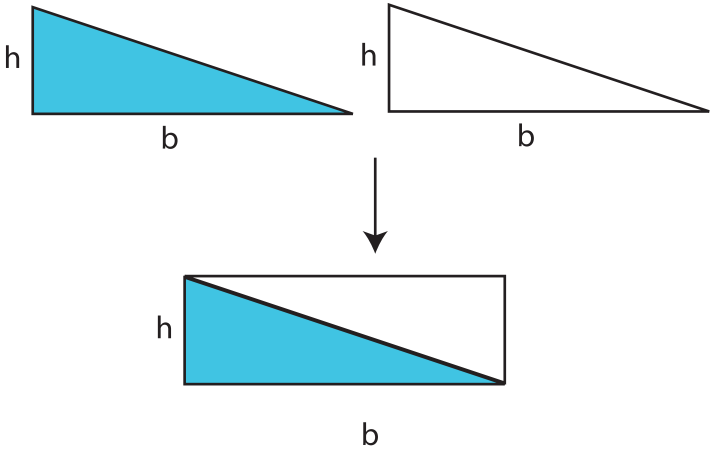
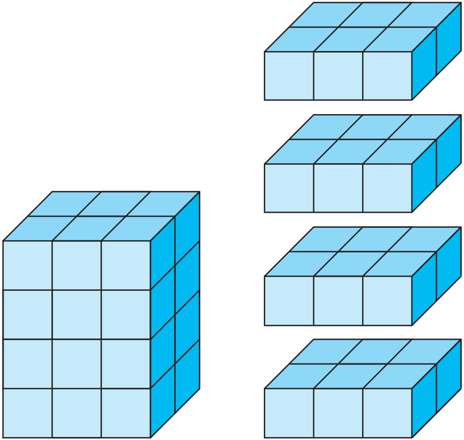


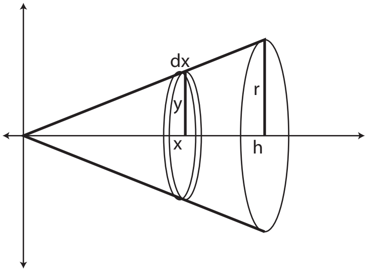
Explaining why the volume formula for prisms is valid
by decomposing into layers



volume = (height)×(area of base)

Decomposing into slices in integral calculus

Derive the formula for the volume of a cone of radius *r* and height *h*



By similar triangles,

$$\frac{y}{x} = \frac{r}{h}$$

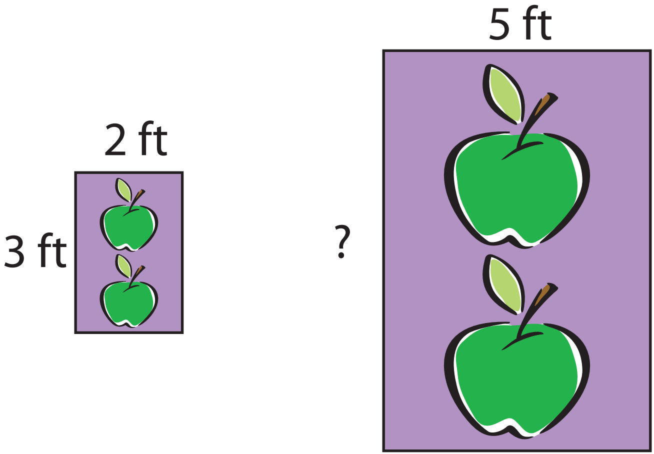
so

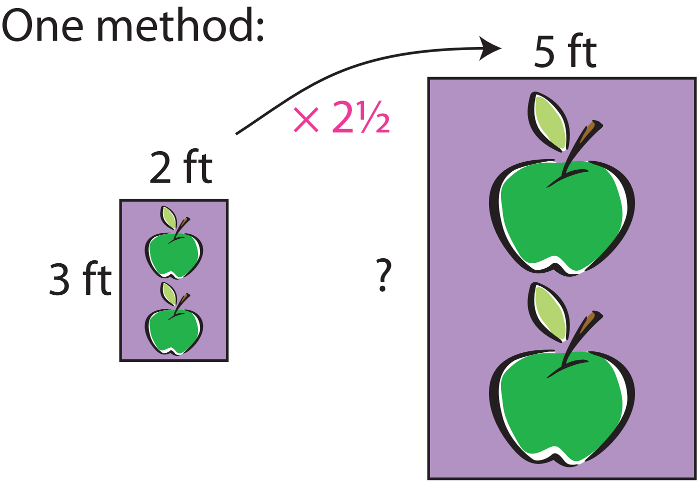
$$\text{Volume} = \int_0^h \pi y^2 dx = \int_0^h \pi \left(\frac{r}{h}\right)^2 x^2 dx = \frac{\pi r^2}{h^2} \cdot \frac{1}{3}x^3\Big]_0^h = \frac{1}{3}\pi r^2 h$$

Reasoning about ratio, proportion, and similarity
in the 6th and 7th grade Focal Points

- 6th: Connecting ratio and rate to multiplication and division
- 7th: Developing an understanding of and applying proportionality, including similarity

Reasoning about similar shapes




One method:
5 ft
× 2½
2 ft
3 ft
?

Reasoning about similar shapes

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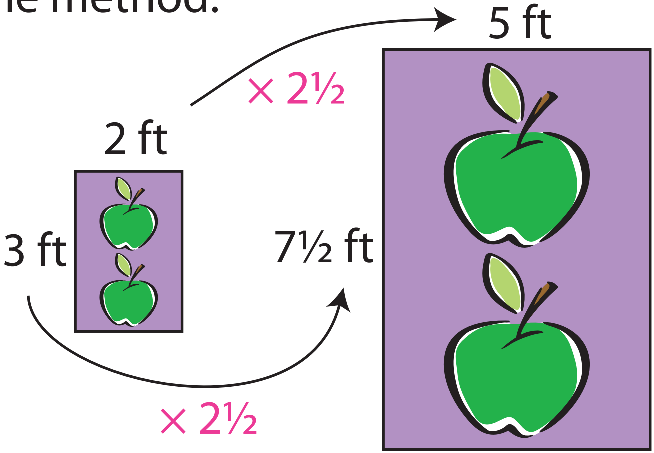

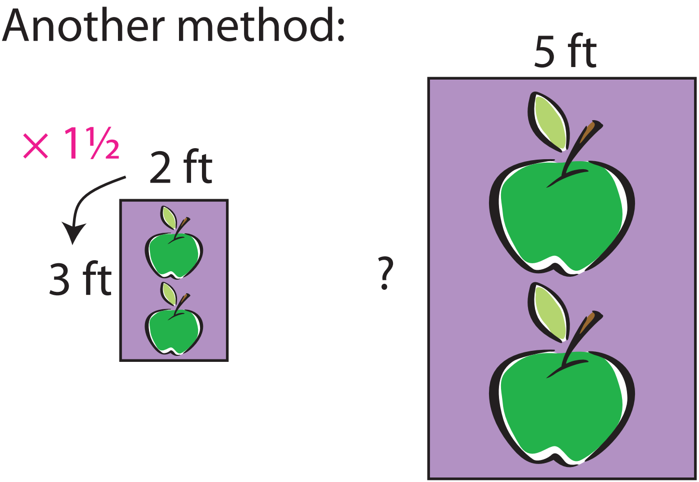
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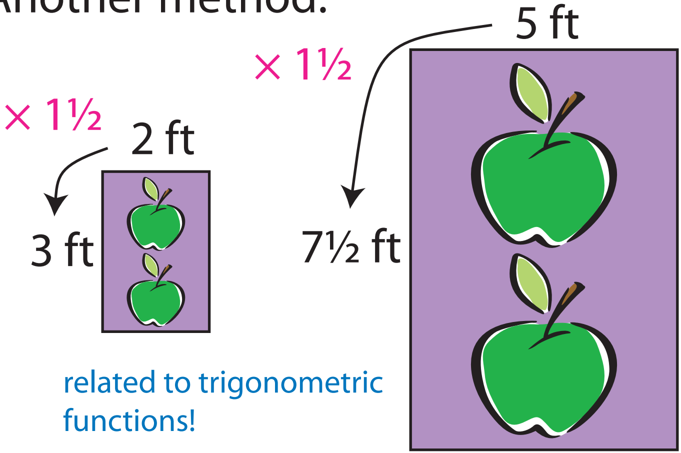

related to trigonometric functions!

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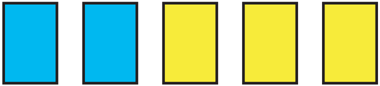
A ratio problem

Blue and yellow paint are mixed in a ratio of 2 to 3 to make green paint. How many pails of blue paint and how many pails of yellow paint will you need to make 30 pails of green paint?

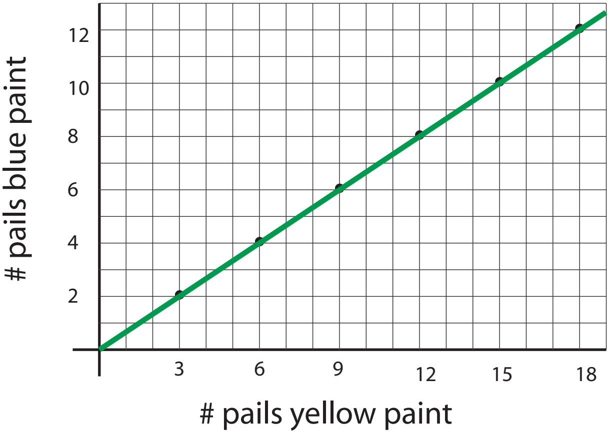
Connecting ratio and rate to multiplication and division

How many pails of each color to make 30 pails of green paint?



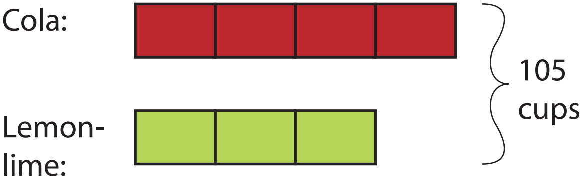
# of batches	1	2	3	4	5	6	7
# pails blue paint	2	4	6	8	10	12	14
# pails yellow paint	3	6	9	12	15	18	21
# pails green paint produced	5	10	15	20	25	**30**	35

Graphing equivalent ratios



Solving ratio problems by reasoning about multiplication and division

Cola and lemon-lime soda are mixed in a ratio of 4 to 3. How much cola and how much lemon-lime soda do you need to make 105 cups of the mixture?



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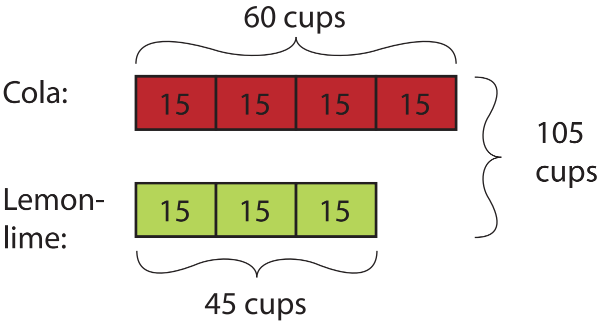

Reasoning about ratio and proportion

- Rows in the multiplication table → equivalent ratios
- Tables of equivalent ratios → tables for functions
- Reasoning about unknown quantities in simple pictures → reasoning about unknown letters

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Reasoning about ratio and proportion

- Rows in the multiplication table $\longrightarrow$ equivalent ratios
- Tables of equivalent ratios $\longrightarrow$ tables for functions
- Reasoning about unknown quantities in simple pictures $\longrightarrow$ reasoning about unknown letters

- Mathematical assertions have reasons
  - *explaining why* is a way of connecting mathematical ideas
  - by *explaining why* we can repeatedly return to fundamental concepts
  - to *explain why* we must think more deeply about the ideas
- Decomposing and composing
  - used *everywhere* in mathematics

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  - meanings of operations
  - place value
  - properties of arithmetic
- Geometry and measurement
  - area and volume — reasoning about decomposing and composing
  - similarity
- Algebra
  - decomposing using properties of arithmetic
  - ratio and proportion — a foundation for linear functions

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