

From *Mathematics for Elementary Teachers*, 2nd edition, by Sybilla Beckmann, copyright © Pearson Education, 2006.

0.1 Ratio and Proportion

There are many situations in daily life as well as in science, medicine, and business that require the use of ratios and proportions. For example, in cooking, if we increase or decrease a recipe, we usually keep the ratios of the various ingredients the same. Any time we use percents, we are using a ratio. When pharmacists mix drugs, they must pay careful attention to the ratios of the ingredients. Before you read on, please do the next class activity.

Class Activity 0A: Comparing Mixtures

In order to answer the questions in the Class Activity 0A, you had to analyze two different ratios. In this section, we will study several different ways to work with ratios—you might have found some of these ways as you thought about the questions in the class activity. Some of these ways of working with ratios involve using only simple, logical reasoning about multiplication and division. Therefore a key point of this section is that if we understand multiplication and division, we can work effectively with ratios. Conversely, working with ratios can provide us with an opportunity to reason about multiplication and division. We will also see that ratios behave like fractions and can be viewed as fractions. In addition to solving a number of problems that involve proportions, we will consider other problems that seem like they might involve a proportion but actually don't.

A ratio describes a specific type of relationship between two quantities. If the ratio of flour to milk in a muffin recipe is 7 to 2, then that means that for every 7 cups of flour you use, you must use 2 cups of milk, as shown in Figure 1. Equivalently, for every 2 cups of milk you use, you must use 7 cups of flour. In general, to say that two quantities are in a **ratio** of A to B means **ratio** (“how many groups? definition) that for every A units of the first quantity there are B units of the second quantity. Equivalently, for every B units of the second quantity there are A units of the first quantity.

There is another slightly different way to think about ratios. Consider again the muffin recipe in which the ratio of flour to milk is 7 to 2. In the last paragraph, we interpreted this ratio as meaning that for every 7 cups of flour

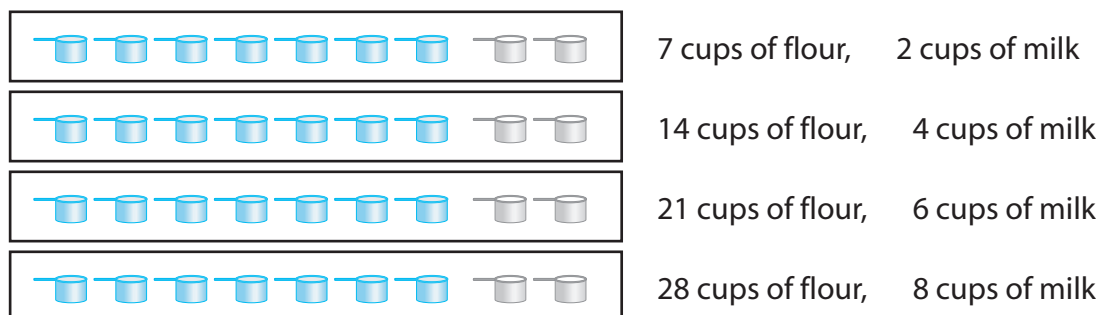


Figure 1: The Ratio 7 to 2 Viewed as Groups of 7 and Groups of 2

you use, you must use 2 cups of milk. Another way to interpret that ratio is that if you divide the flour into 7 parts of equal volume, you must use 2 parts of that same volume of milk, as shown in Figure 2. For example, if you



Figure 2: The Ratio 7 to 2 Viewed as 7 Groups and 2 Groups

use 7 pints of flour, you should use 2 pints of milk; if you use 7 pails of flour, you should use 2 pails of milk. No matter what the size of the container you use, if you use 7 containers worth of flour, you should use 2 containers worth of milk. In general, with this point of view, to say that two quantities are in a **ratio** of A to B means that if we divide the first quantity into A equal parts, then the second quantity consists of B parts of the same size. Notice that in the case of the muffin recipe, “parts of the same size” meant that the parts were of equal *volume*. In other contexts, “parts of the same size” could mean that the parts are of equal weight or equal length, for example.

The two different points of view about ratios correspond to the two dif-

ratio (“how many in each group?” definition)

ferent points of view about division. The first view of ratio corresponds to the “how many groups?” view of division because we view the quantities as consisting of multiple groups of the same size (such as groups of 7 cups of flour and groups of 2 cups of milk). The second view of ratio corresponds to the “how many in each group?” view of division because we put the quantities into a fixed number of groups of the same size (such as 7 groups of flour and 2 groups of milk, all groups being of the same size).

A **rate** is a ratio between two quantities that are measured in different units. For example, **speed**, which is the ratio of distance traveled to elapsed time, is a rate because distance is measured in miles, kilometers, or other similar units, whereas time is measured in hours, minutes, or seconds. If you are traveling at a speed of 60 mph, that means that you are traveling 60 miles for every hour that you travel. When we work with rates, we use the first point of view about ratios. **rate**

To indicate a ratio we can use words, as in

the ratio of flour to milk is 7 to 2,

we can use a colon, as in

the ratio of flour to milk is 7 : 2,

or we can use a fraction, as in

the ratio of flour to milk is $7/2$ (or $\frac{7}{2}$).

Later in this section we will see why it makes sense to write ratios as fractions.

Before you read on, do the next two class activities in order to use the two different points of view about ratios to solve problems. In the first activity you will use the first point of view (the “how many groups?” view) when you work with ratio tables. In the second activity, you will use the second point of view (the “how many in each group?” view) when you work with strip diagrams. The sixth grade mathematics textbooks used in Singapore nicely illustrate the use of strip diagrams for solving ratio problems (see [?], volume 6A).

Class Activity 0B: (c) Using Ratio Tables

Class Activity 0C: (c) Using Strip Diagrams to Solve Ratio Problems

Equivalent Ratios, Proportions, and Ratio Tables

Suppose that to make 1 batch of muffins we need 7 cups of flour and 2 cups of milk. If we make several batches of muffins, the ratio of total flour used to total milk used is still 7 to 2 (because for every 7 cups of flour we use 2 cups of milk). On the other hand, the actual amounts of flour and milk that are needed describe a different ratio, such as 14 to 4 (if we make 2 batches). Thus the ratios 7 to 2 and 14 to 4 are **equivalent**, in other words, they are the same ratio. We may write this statement about equivalent ratios as the following equation:

$$\frac{7}{2} = \frac{14}{4}$$

equivalent ratios

proportion

The statement that two ratios are equivalent is a **proportion**. So the previous equation is a proportion.

An easy way to begin to work with ratios and proportions is to use a ratio table. A ratio table is just a table that lists equivalent ratios. If we make several batches of our muffins or half a batch of muffins we will need the amounts of flour and milk shown in the ratio table of Table 1. How do

# of batches	1	2	3	4	5	$\frac{1}{2}$	$2\frac{1}{2}$	N
# cups flour	7	14	21	28	35	$3\frac{1}{2}$	$17\frac{1}{2}$	$N \cdot 7$
# cups milk	2	4	6	8	10	1	5	$N \cdot 2$

Table 1: A Ratio Table for Batches of Muffins

we determine the entries in this table?

At first, we might start filling in the table by simply adding 7 cups of flour and 2 cups of milk every time we make an additional batch, as indicated in Table 2 and Figure 3. But then we might notice that the entries in Table 3 are entries in the multiplication table, as in Figure 4. In fact, even the entries for $\frac{1}{2}$ of a batch, $2\frac{1}{2}$ batches, and N batches can be viewed as part of an “extended” multiplication table.

Thus we can use simple reasoning about multiplication to determine the entries in Table 1. For example, to make 4 batches of muffins we will need 4 times as much flour and 4 times as much milk as in 1 batch. To make $\frac{1}{2}$

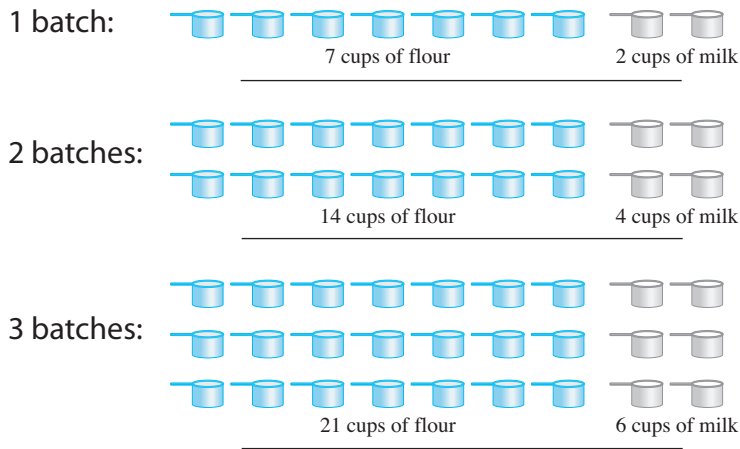


Figure 3: Different Combinations of Flour and Milk That Are in the Ratio 7 to 2

×	1	2	3	4	5	6	7	8	9	10	...	$2\frac{1}{2}$
1	1	2	3	4	5	6	7	8	9	10		
2	2	4	6	8	10	12	14	16	18	20		5
3	3	6	9	12	15	18	21	24	27	30		
4	4	8	12	16	20	24	28	32	36	40		
5	5	10	15	20	25	30	35	40	45	50		
6	6	12	18	24	30	36	42	48	54	60		
7	7	14	21	28	35	42	49	56	63	70		$17\frac{1}{2}$
⋮												

Figure 4: Entries in a Ratio Table Come From an Extended Multiplication Table

# batches:	1		2		3		4		5
# cups flour:	7	$\xrightarrow{+7}$	14	$\xrightarrow{+7}$	21	$\xrightarrow{+7}$	28	$\xrightarrow{+7}$	35
# cups milk:	2	$\xrightarrow{+2}$	4	$\xrightarrow{+2}$	6	$\xrightarrow{+2}$	8	$\xrightarrow{+2}$	10

Table 2: Finding Entries in a Ratio Table by Repeatedly Adding

of a batch of muffins we will need $\frac{1}{2}$ as much flour and $\frac{1}{2}$ as much milk, so we'll need $\frac{1}{2} \cdot 7 = 3\frac{1}{2}$ cups of flour and $\frac{1}{2} \cdot 2 = 1$ cup of milk. In general, to make N batches of muffins, we will need N times as much flour and milk as in 1 batch, so we will need $N \cdot 7$ cups of flour and $N \cdot 2$ cups of milk. This reasoning is summarized in Table 3.

# batches:	1		4		1		$\frac{1}{2}$		1		N
# cups flour:	7	$\xrightarrow{\times 4}$	28		7	$\xrightarrow{\times \frac{1}{2}}$	$3\frac{1}{2}$		7	$\xrightarrow{\times N}$	$N \cdot 7$
# cups milk:	2	$\xrightarrow{\times 4}$	8		2	$\xrightarrow{\times \frac{1}{2}}$	1		2	$\xrightarrow{\times N}$	$N \cdot 2$

Table 3: Finding Entries in a Ratio Table by Multiplying

In N batches of muffins the ratio of flour to milk is still 7 to 2 because for every 7 cups of flour, we use 2 cups of milk, no matter what N is (as long as it's positive). But on the other hand, if we make N batches of muffins, we will use $N \cdot 7$ cups of flour and $N \cdot 2$ cups of milk, so we can also describe the ratio of flour to milk as $N \cdot 7$ to $N \cdot 2$. Thus the two ratios are equivalent, so that we have the following proportion

$$\frac{7}{2} = \frac{N \cdot 7}{N \cdot 2}$$

In general, the preceding reasoning tells us that the ratio A to B (where A and B are positive) is equivalent to the ratio $N \cdot A$ to $N \cdot B$, so that

$$\frac{A}{B} = \frac{N \cdot A}{N \cdot B}$$

is a proportion (if N is positive). Notice that we had made essentially this same statement about equivalent fractions in Section 3.3.

The previous paragraph tells us that we can create an equivalent ratio by multiplying the entries in a ratio by the same number positive N (which does not need to be a whole number—it can even be a fraction or a decimal).

There is another useful way to create equivalent ratios. Let's consider a shade of orange paint created by mixing red paint with yellow paint in a ratio of 1 to 3, as shown in Figure 5. If we use 1 container worth of red paint, then

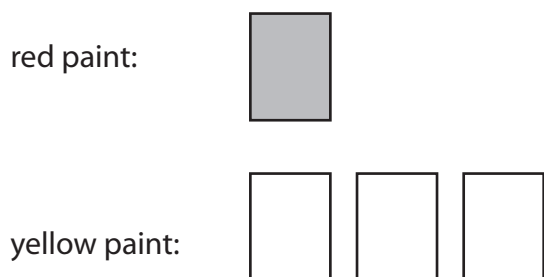


Figure 5: Orange Paint Created by Mixing Yellow and Red Paint in a Ratio of 3 to 1

we must use 3 containers worth of yellow paint to make this shade of orange (where all containers are the same size). This means that however much red paint we use, we must use 3 times as much yellow paint and however much yellow paint we use, we must use $\frac{1}{3}$ as much red paint in order to create equivalent ratios of red to yellow paint. This reasoning is summarized in Table 4.

# containers red:	1	N	1	$\frac{1}{3} \cdot M$
	$\downarrow \times 3$	$\downarrow \times 3$	$\uparrow \times \frac{1}{3}$	$\uparrow \times \frac{1}{3}$
# containers yellow:	3	$3 \cdot N$	3	M

Table 4: Relating Amounts of Yellow and Red Paint When the Ratio of Yellow to Red Paint is 3 to 1

In general, the preceding reasoning tells us that the ratio A to $N \cdot A$ is equivalent to the ratio B to $N \cdot B$ (if A , B , and N are positive). In fraction form we can write this as

$$\frac{A}{N \cdot A} = \frac{B}{N \cdot B} \quad \text{or} \quad \frac{N \cdot A}{A} = \frac{N \cdot B}{B}$$

Returning to the orange paint created by mixing yellow and red paint in a ratio of 3 to 1, if we want to make the orange paint using $2\frac{1}{2}$ gallons of red paint, then we should use

$$3 \cdot 2\frac{1}{2} = 7\frac{1}{2}$$

gallons of yellow paint because we use 3 times as much yellow paint as red paint. If we want to make the orange paint using 2 gallons of yellow paint, then we should use

$$\frac{1}{3} \cdot 2 = \frac{2}{3}$$

gallons of red paint because we use $\frac{1}{3}$ as much red paint as yellow paint. This reasoning is summarized in Table 5.

# gallons red:	1	$2\frac{1}{2}$	1	$\frac{2}{3}$
	$\downarrow \times 3$	$\downarrow \times 3$	$\uparrow \times \frac{1}{3}$	$\uparrow \times \frac{1}{3}$
# gallons yellow:	3	$7\frac{1}{2}$	3	2

Table 5: Finding Amounts of Yellow or Red Paint When the Ratio of Yellow to Red Paint is 3 to 1

Notice that we could use the reasoning we used for Table 3 in this case as well, as indicated in Table 6.

# gallons yellow:	3	$\xrightarrow{\times 2\frac{1}{2}}$	$7\frac{1}{2}$	3	$\xrightarrow{\times \frac{2}{3}}$	2
# gallons red:	1	$\xrightarrow{\times 2\frac{1}{2}}$	$2\frac{1}{2}$	1	$\xrightarrow{\times \frac{2}{3}}$	$\frac{2}{3}$

Table 6: Finding Amounts of Yellow or Red Paint When the Ratio of Yellow to Red Paint is 3 to 1

Class Activity 0D: Using Simple Reasoning to Find Equivalent Ratios and Rates

We will now apply the reasoning about ratios and multiplication that we have been studying in order to solve proportions in several different ways.

Solving Proportions with Multiplication, Division, and Simple Logical Reasoning

When you think of solving proportions, you may think of the method in which you set two fractions equal to each other, cross-multiply these fractions, and then solve the resulting equation. However, we can also solve proportions by using multiplication, division, and simple logical reasoning. When students solve proportion problems this way they have an opportunity to think more deeply about the operations of multiplication and division and when to apply these operations. Perhaps this is why the mathematics textbooks used in Singapore at 6th grade (see [?], volume 6A) do not teach the “cross-multiplication” method but instead have children solve proportion problems by reasoning about multiplication and division.

We will now study two simple methods for solving proportions that arise in problems.

Problem: Mr. Arias used 4 gallons of gas to drive 100 miles. Assuming that Mr. Arias continues to get the same gas mileage (in other words, assuming that the rate of miles driven to gas used continues to remain the same), how far will Mr. Arias be able to drive with 20 gallons of gas?

“Direct” Solution Method: Since 20 gallons of gas is 5 times as much as 4 gallons of gas, Mr. Arias will be able to drive 5 times as far, so he will be able to drive $5 \times 100 = 500$ miles. This line of reasoning is summarized in Table 7 (in both tabular and fraction form) and shown pictorially with the “double number line” in Figure 6.

“Going Through 1” Solution Method: If we think of dividing the 100 miles equally among the 4 gallons of gas, we see that each gallon of gas will take Mr. Arias 25 miles. So the number of miles that Mr. Arias can drive is always 25 times the number of gallons of gas he has. Since Mr. Arias has 20 gallons of gas, he can drive $25 \times 20 = 500$ miles. This line of reasoning is summarized in two different ways in Tables 8 and 9 (in both tabular and fraction forms) and is indicated pictorially with the double number line in Figure 7.

	problem:	1st step:		2nd step:
miles:	100 ?	100 ?		100 $\xrightarrow{\times 5}$ 500
gallons:	4 20	4 $\xrightarrow{\times 5}$ 20		4 $\xrightarrow{\times 5}$ 20

$\frac{100}{4} = \frac{?}{20}$	$\frac{100}{4} = \frac{?}{20}$ $\xrightarrow{\times 5}$	$\frac{100}{4} = \frac{500}{20}$ $\xrightarrow{\times 5}$
--------------------------------	--	--

Table 7: Solving a Proportion by Using Reasoning about Multiplication and Division

Class Activity 0E: Solving Proportions with Multiplication and Division

In the problem about Mr. Arias’s gas mileage, the numbers were easy to work with. What if the numbers in a problem are not as nice? We can still use the same reasoning, but the “going through 1” method may seem easier in this case.

Problem: If you want to use 4 cups of flour in a muffin recipe in which the ratio of flour to milk is 7 to 2, how much milk should you use?

“Going Through 1” Solution Method: We first ask how much milk we would need for 1 cup of flour. Once we know how much milk we would need for 1 cup of flour, we’ll multiply that amount by 4 to find out how much milk we’ll need for 4 cups of flour. If 7 cups of flour take 2 cups of milk, then we can think of the 2 cups of milk as divided equally among the 7 cups of flour. Therefore, according to the “how many in each group?” view of division, each cup of flour takes

$$2 \div 7 = \frac{2}{7}$$

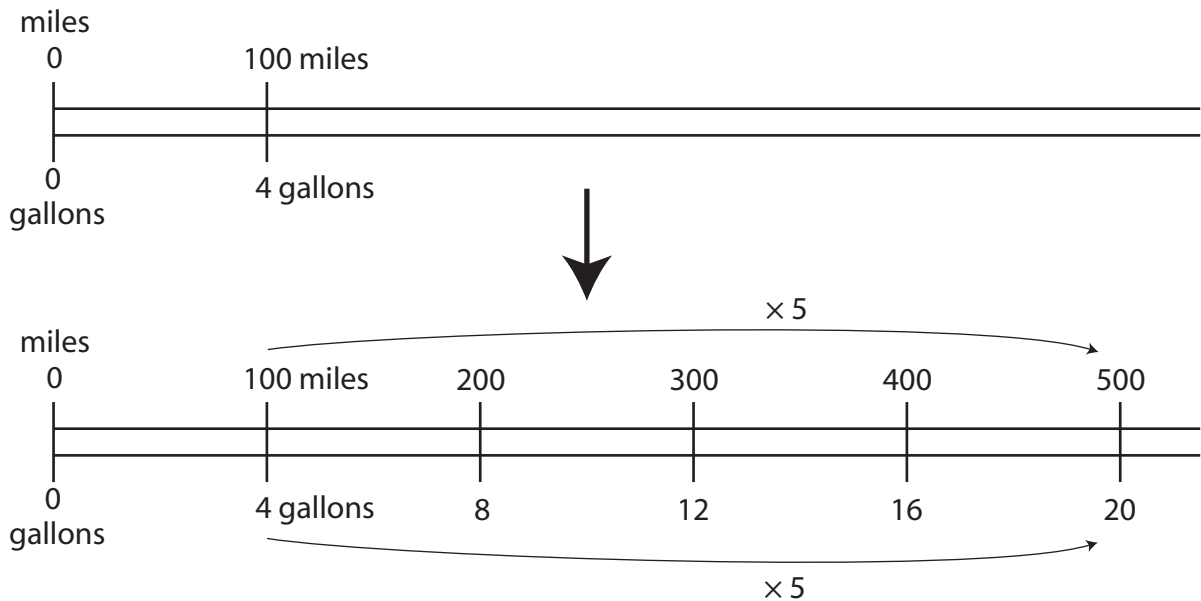


Figure 6: Solving a Proportion by Using Reasoning about Multiplication and Division on a Double Number Line

of a cup of milk. Now if we want to use 4 cups of flour, then we need to use 4 groups of $\frac{2}{7}$ cup of milk. In other words, we need to use

$$4 \cdot \frac{2}{7} = \frac{8}{7} = 1\frac{1}{7}$$

cups of milk. Table 10 summarizes this reasoning.

“Direct” Solution Method: If we wanted to use 21 cups of flour instead of 4 cups of flour, the solution would be easy: 21 cups is 3 times as much as 7 cups (because $21 \div 7 = 3$), so we would also need to use 3 times as much milk, namely $3 \cdot 2 = 6$ cups milk. The top portion of Table 11 summarizes this reasoning. The same reasoning applies to 4 cups of flour: 4 cups is $\frac{4}{7}$ times as much as 7 cups (because $4 \div 7 = \frac{4}{7}$), so we also need to use $\frac{4}{7}$ times as much milk, namely $\frac{4}{7} \cdot 2 = 1\frac{1}{7}$ cups of milk. The bottom portion of Table 11 summarizes this reasoning.

“Direct” Solution Method, Using “Parts”: Another way to solve the problem is to think about the ratio 7 to 2 of flour to milk in terms of parts.

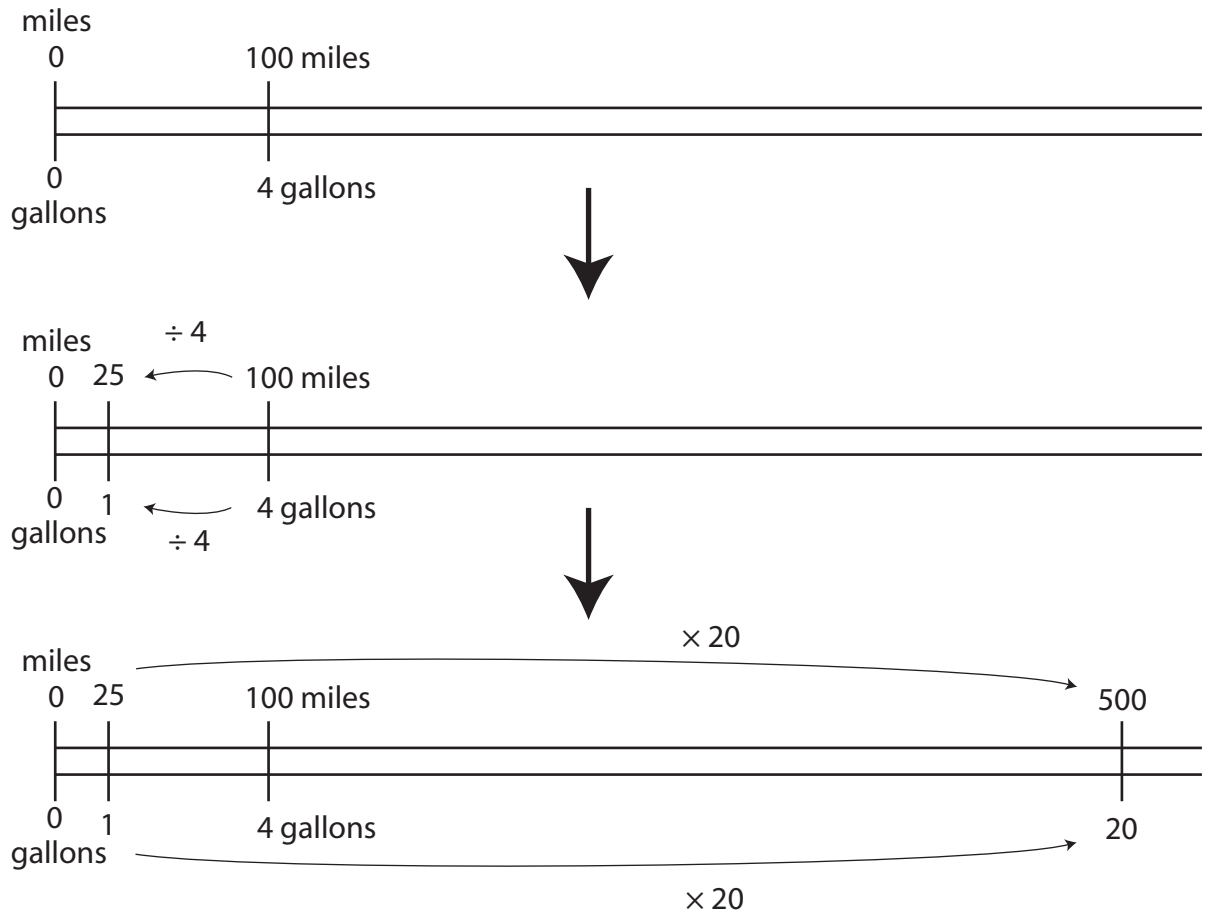


Figure 7: Solving a Proportion by Using Reasoning about Multiplication and Division on a Double Number Line, "Going Through 1" Method

	problem:	1st step:	2nd step:
miles:	100 ?	100 ?	100 500
		↑ × 25	↑ × 25 ↑ × 25
gallons:	4 20	4 20	4 20

$$\frac{100}{4} = \frac{?}{20} \quad \times 25 \left(\frac{100}{4} = \frac{?}{20} \right) \quad \times 25 \left(\frac{100}{4} = \frac{500}{20} \right) \times 25$$

Table 8: Solving a Proportion by Using Reasoning about Multiplication and Division

There are 7 parts of flour and 2 parts of milk. If 7 parts of flour are 4 cups, then each part is $4 \div 7 = \frac{4}{7}$ of a cup (since the 4 cups are divided equally among the 7 parts). The 2 parts of milk are therefore $2 \cdot \frac{4}{7} = 1\frac{1}{7}$ cups of milk. This reasoning is summarized in the strip diagram in Figure 8.

Ratios and Fractions

We have been writing ratios as fractions, but we defined ratios and fractions in different ways. So why is it legitimate to equate ratios and fractions as we have been doing?

Before you read on, try to relate the ratios in the next class activity to fractions, where the fractions are viewed either as parts of wholes, or as solutions to division problems.

Class Activity 0E: Ratios, Fractions, and Division

Let's now see how every ratio of quantities has a naturally associated fraction (viewed as parts of a whole). Consider a bread recipe in which the ratio of flour to water is 14 to 5, so that if we use 14 cups of flour, we will need 5 cups of water. If we think of dividing those 14 cups of flour equally among the 5 cups of water, then, according to the meaning of division (the

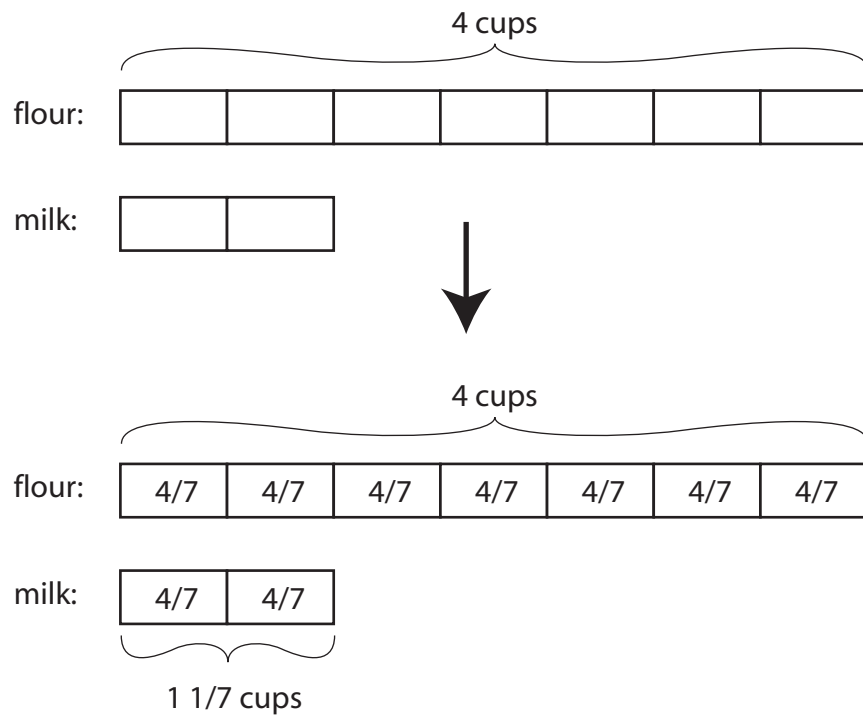


Figure 8: Solving a Proportion by Using Reasoning about Multiplication and Division, "Direct Method", Using "Parts"

	problem:		solution:		
miles:	100	?	100	$\xrightarrow{\div 4}$ 25	$\xrightarrow{\times 20}$ 500
gallons:	4	20	4	$\xrightarrow{\div 4}$ 1	$\xrightarrow{\times 20}$ 20

$$\frac{100}{4} = \frac{?}{20}$$

$$\frac{100}{4} \xrightarrow{\div 4} \frac{25}{1} \xrightarrow{\times 20} \frac{500}{20}$$

Table 9: Solving a Proportion by Using Reasoning about Multiplication and Division, “Going Through 1” Method

	problem:		solution:		
cups flour:	7	4	7	$\xrightarrow{\div 7}$ 1	$\xrightarrow{\times 4}$ 4
cups milk:	2	?	2	$\xrightarrow{\div 7}$ $\frac{2}{7}$	$\xrightarrow{\times 4}$ $1\frac{1}{7}$

Table 10: Solving a Proportion by Using Reasoning about Multiplication and Division and “Going Through 1”

“how many in each group?” view), each cup of water goes with

$$14 \div 5 = \frac{14}{5}$$

cups of flour. So in this recipe, there are $\frac{14}{5}$ cups of flour per cup of water. In this way, the ratio 14 to 5 of flour to water is naturally associated with the fraction $\frac{14}{5}$, which tells us the number of cups of flour per cup of water.

In general, a ratio A to B relating two quantities gives rise to the fraction $\frac{A}{B}$, which tells us the number of units of the first quantity there are per unit amount of the second quantity. (The one exception, which doesn’t even occur in ordinary circumstances, is when B is zero—in that case we cannot form the fraction $\frac{A}{B}$.)

Similarly, a fraction (or other number) that tells us how much of one quantity there is per unit amount of another quantity, gives rise to a ratio

easier problem:	solution:
cups flour: 7 21	7 $\xrightarrow{\times 3}$ 21
cups milk: 2 ?	2 $\xrightarrow{\times 3}$ 6
our problem:	solution:
cups flour: 7 4	7 $\xrightarrow{\times \frac{4}{7}}$ 4
cups milk: 2 ?	2 $\xrightarrow{\times \frac{4}{7}}$ $1\frac{1}{7}$

Table 11: Solving a Proportion by Using Reasoning about Multiplication and Division, “Direct Method”

between the two quantities. If we have a paint mixture in which we use $\frac{2}{3}$ cup of blue paint per cup of yellow paint, then in this mixture, the ratio of blue paint to yellow paint is

$$\frac{2}{3} \text{ to } 1$$

If we want to make the same shade of green paint using 3 cups of yellow paint, then we should use 3 times as much blue paint, or

$$3 \cdot \frac{2}{3} = 2$$

cups of blue paint. Therefore, we can also express the ratio of blue paint to yellow paint in the mixtures as

$$2 \text{ to } 3$$

In general, if there is $\frac{A}{B}$ of one quantity per unit amount of a second quantity, then the ratio of the first quantity to the second is

$$\frac{A}{B} \text{ to } 1$$

which is equivalent to the ratio

$$B \cdot \frac{A}{B} \text{ to } B \cdot 1$$

because in the latter ratio we are using B times as much of both quantities as in the former ratio. But the previous ratio, $B \cdot \frac{A}{B}$ to $B \cdot 1$, is just

A to B

So we conclude that

$\frac{A}{B}$ to 1 is equivalent to A to B

Therefore, according to the previous paragraph, the fraction

$\frac{A}{B}$

gives rise to the ratio

A to B

So the ratio A to B gives rise to the fraction $\frac{A}{B}$ and the fraction $\frac{A}{B}$ gives rise to the ratio A to B and therefore we may equate ratios and fractions.

The Logic Behind Solving Proportions by Cross-Multiplying Fractions

You are probably familiar with the technique of solving proportions by cross-multiplying. Why is this a valid technique for solving proportions? We will examine this now. Before you read on, please do the next Class Activity.

Class Activity 0E: Solving Proportions by Cross-Multiplying Fractions

Why is the method of setting of solving proportions by setting two fractions equal to each other and cross-multiplying valid? Consider a light blue paint mixture made with $\frac{1}{4}$ cup blue paint and 4 cups white paint. How much blue paint will you need if you want to use 6 cups white paint and if you are using the same ratio of blue paint to white paint in order to make the same shade of light blue paint? A common method for solving such a problem is to set up the following proportion in fraction form:

$$\frac{\frac{1}{4}}{4} = \frac{A}{6}$$

Here, A represents the as yet unknown amount of blue paint you will need for 6 cups of white paint. We then cross-multiply to get

$$6 \cdot \frac{1}{4} = 4 \cdot A$$

Therefore,

$$A = \left(6 \cdot \frac{1}{4}\right) \div 4 = 1\frac{1}{2} \div 4 = \frac{3}{8}$$

so that you must use $\frac{3}{8}$ cups of blue paint for 6 cups of white paint.

Let's analyze the preceding steps. First, when we set the two fractions

$$\frac{1}{4}$$

and

$$\frac{A}{6}$$

equal to each other, why can we do that and what does it mean? If we think of the fractions as representing division—that is,

$$\frac{1}{4} \div 4$$

and

$$A \div 6$$

then each of these expressions stands for the number of cups of blue paint per cup of white paint. We want to use the same amount of blue paint per cup of white paint either way; therefore, the two fractions should be equal to each other, or

$$\frac{1}{4} = \frac{A}{6} \tag{1}$$

Next, why do we cross-multiply? We can cross-multiply because two fractions are equal exactly when their “cross-multiples” are equal. Recall that the method of cross-multiplying is really just a shortcut for giving fractions a common denominator. If we give the fractions in Equation 1 the common denominator $6 \cdot 4$, which is the product of the two denominators, then we can replace Equation 1 with the proportion

$$\frac{\frac{1}{4} \cdot 6}{4 \cdot 6} = \frac{A \cdot 4}{6 \cdot 4} \tag{2}$$

In terms of the paint mixture, both sides of this proportion now refer to 24 cups of paint, instead of 4 cups and 6 cups of paint, as in Equation 1. But two fractions that have the same denominator are equal exactly when their numerators are equal. Since the denominators of the fractions in Equation 2 are equal, since $4 \cdot 6 = 6 \cdot 4$, the proportion will be solved exactly when the numerators are equal, namely when

$$\frac{1}{4} \cdot 6 = A \cdot 4 \quad (3)$$

Therefore we can solve the proportion in Equation 1 by solving Equation 3, which was obtained by cross-multiplying.

Class Activity 0E: (c) Can You Always Use a Proportion?

(+) Using Proportions: The Consumer Price Index

At some point you have probably heard older people wistfully recall the lower prices of days gone by, as in, “When I was young, a candy bar only cost a nickel!” As time goes by, most items become more expensive, due to inflation. (Computers are a notable exception.) So in years past, one dollar bought more than a dollar buys now. But in years past, most people also earned less than they do now. So, what is a fair way to compare costs of items across years? What is a fair way to compare wages earned in different years? The standard way to make such comparisons is with the Consumer Price Index, which is determined by the Bureau of Labor Statistics of the U.S. Department of Labor. To use the Consumer Price Index, we must work with proportions.

According to the Bureau of Labor Statistics, the Consumer Price Index (CPI) is a measure of the average change over time in the prices paid by urban consumers for a market basket of consumer goods and services. The CPI market basket is constructed from detailed expenditure information provided from surveys of families and individuals on what they actually bought. The CPI measures inflation as experienced by consumers in their day-to-day living expenses. This, and other information about the CPI, is available on the Bureau of Labor Statistics website. See www.aw-bc.com/beckmann.

Here is an example of how to use the CPI to compare salaries in different years. According to the Bureau of Labor Statistics, the average hourly rate

for elementary school teachers was \$28.79 in 2000 and \$32.46 in 2004. Now, the average hourly rate was higher in 2004 than it was in 2000, but most items were also more expensive in 2004 than they were in 2000. So, in 2004, did elementary school teachers have more “buying power” than they did in 2000 or not? In other words, did elementary school teachers’ hourly rate go up faster than inflation or not? We can use the CPI to determine the answer to this question.

The CPI assigns a number to each year (in recent history). Table 12 shows the CPI for some selected years.

The Consumer Price Index (selected years)							
year	CPI	year	CPI	year	CPI	year	CPI
1940	14.0	1990	130.7	1995	152.4	2000	172.2
1950	24.1	1991	136.2	1996	156.9	2001	177.1
1960	29.6	1992	140.3	1997	160.5	2002	179.9
1970	38.8	1993	144.5	1998	163.0	2003	184.0
1980	82.4	1994	148.2	1999	166.6	2004	188.9

Table 12: The Consumer Price Index for Selected Years

We interpret these numbers as follows: Because the CPI was 172.2 in 2000 and 188.9 in 2004, this means that for every \$172.20 you would have spent toward a “market basket of consumer goods and services” in 2000, you would have had to spend \$188.90 in the year 2004. We can use this to compare the teachers’ hourly rates in 2000 and 2004. Think of each group of \$172.20 as buying a “unit” of a market basket of goods and services in 2000. So in 2000, on average an elementary school teacher could buy

$$28.79 \div 172.20 = 0.167189\dots$$

units of a market basket of goods and services with 1 hour’s earnings. To buy the same number of units of a market basket of goods and services in

2000, namely, 0.167189... units, you would have needed

$$0.167189... \times \$188.90 = \$31.58$$

because each unit of a market basket of goods and services cost \$188.90 in 2004. In other words, \$31.58 in 2004 had the same buying power as did \$28.79 in 2000. Given that in the year 2004 elementary school teachers made an average of \$32.46 per hour, which is more than \$31.58, elementary school teachers had more buying power in 2004 than they did in 2000. Thus, from 2000 to 2004, elementary school teachers' hourly rate went up faster than inflation.

Here is another way to compare elementary school teachers' average hourly rates in 2000 and 2004 using the CPI. Because the CPI was 172.2 in 2000 and 188.9 in 2004, we can again think of \$172.20 in 2000 and \$188.90 in 2004 as buying a "unit" of a market basket of consumer goods and services. Now,

$$188.90 \div 172.20 = 1.096980...$$

Therefore, a "unit" of a market basket of goods and services cost 1.096980... times as much in 2004 as it did in 2000. Hence, an hourly rate of \$28.79 in 2000 had the same buying power as did an hourly rate 1.096980... times as much, namely,

$$1.096980... \times \$28.79 = \$31.58$$

in 2004. We reach the same conclusion as in the previous paragraph.

Either of these calculations is called **adjusting for inflation**. In both calculations, we converted \$28.79, the average hourly rate for elementary teachers in 2000, to 2004 dollars, in which it became \$31.58. So, a 2000 hourly rate of \$28.79 had the same buying power as an hourly rate of \$31.58 did in 2004. Notice that another way to set up the calculations to adjust for inflation would be to set up either of the following two proportions:

$$\frac{28.79}{172.2} = \frac{x}{188.9}$$

or

$$\frac{188.9}{172.2} = \frac{x}{28.79}$$

Here, x stands for the dollar amount in 2004 that had the same buying power as did \$28.79 in 2000.

Class Activity 0BB: (+) The Consumer Price Index p. 207**Practice Problems for Section 0.1**

1. A soda mixture can be made by mixing cola and lemon-lime soda in a ratio of 4 to 3. Explain how to solve each of the following problems about the soda mixture by using logical reasoning about multiplication and division in two ways: with the aid of a ratio table and with the aid of a strip diagram.
 - (a) How much cola and how much lemon-lime soda should you use to make the soda mixture with 48 cups of cola? How much soda mixture will this make?
 - (b) How much cola and how much lemon-lime soda should you use to make 105 cups of the soda mixture?
2. Traveling at a constant speed a race car goes 15 miles every 6 minutes. Use simple, logical reasoning supported by a double number line to help you determine the answers to the next questions.
 - (a) How far does the race car go in 15 minutes?
 - (b) How long does it take the race car to go 20 miles?
3. The ratio of Quint's CDs to Chris's CDs was 7 to 3. After Quint gave 6 CDs to Chris, they had an equal number of CDs. How many CDs did Chris have at first? Explain how to solve this problem with the aid of a strip diagram.
4. Cali mixed $3\frac{1}{2}$ cups of red paint with $4\frac{1}{2}$ cups of yellow paint to make an orange paint. How many cups of red paint and how many cups of yellow paint will Cali need to make 12 cups of the same shade of orange paint? Solve this problem using only simple reasoning about multiplication or division (or both). Explain your reasoning.
5. To make a punch you mixed $\frac{1}{4}$ cup grape juice concentrate with $1\frac{1}{2}$ cups bubbly water. If you want to make the same punch using 2 cups of bubbly water, then how many cups of grape juice concentrate should you use? Solve this problem using only simple reasoning about multiplication or division (or both). Explain your reasoning.

6. In order to reconstitute a medicine properly, a pharmacist must mix 10 milliliters (mL) of a liquid medicine for every 12 mL of water. If one dose of the medicine/water mixture must contain $2\frac{1}{2}$ mL of medicine, then how many milliliters of medicine/water mixture provides one dose of the medicine? Solve this problem using only simple reasoning about multiplication or division (or both). Explain your reasoning.
7. Jose mixed 3 cups of blue paint with 4 cups of red paint to make a purple paint. For each of the following fractions and division problems, interpret the fraction or the division problem in terms of Jose's paint mixture and explain why your interpretation makes sense:

$$\frac{3}{7} \text{ or } 3 \div 7; \quad \frac{4}{7} \text{ or } 4 \div 7; \quad \frac{3}{4} \text{ or } 3 \div 4;$$

$$\frac{4}{3} \text{ or } 4 \div 3; \quad \frac{7}{3} \text{ or } 7 \div 3; \quad \frac{7}{4} \text{ or } 7 \div 4$$

8. Which of the following two mixtures will be more salty?
- 3 tablespoons of salt mixed in 4 cups of water
 - 4 tablespoons of salt mixed in 5 cups of water

Solve this problem in two different ways, explaining why you can solve the problem the way you do.

9. Suppose that a logging crew can cut down five acres of trees every two days. Assume that the crew works at a steady rate. Solve problems (a) and (b) using logical thinking and using the most elementary reasoning you can. Explain your reasoning clearly.
- (a) How many days will it take the crew to cut 8 acres of trees? Give your answer as a mixed number.
- (b) Now suppose there are *three* logging crews that all work at the same rate as the original one. How long will it take these three crews to cut down 10 acres of trees?
10. If 3 people take 2 days to paint 5 fences, how long will it take 2 people to paint 1 fence? (Assume that the fences are all the same size and the painters are all equally good workers, and work at a steady rate.) Can

this problem be solved by setting up the following proportion to find how long it will take 2 people to paint 5 fences?

$$\frac{3 \text{ people}}{2 \text{ days}} = \frac{2 \text{ people}}{x \text{ days}}$$

Solve the problem by thinking logically about the situation. Explain your reasoning clearly.

11. (+) If a candy bar cost a nickel in 1960, then how much would you expect to have to pay for the same size and type of candy bar in 2000, according to the CPI?
12. (+)
 - (a) What amount of money had the same buying power in 1950 as \$50,000 did in 2000?
 - (b) What amount of money had the same buying power in 1980 as \$50,000 did in 2000?
13. (+) If an item cost \$5.00 in 1990 and \$6.50 in 2000, did its price go up faster, slower, or the same as inflation?
14. (+) Suppose that from 1990 to 2000 the price of a 16-ounce box of brand A cereal went from \$2.39 to \$3.89. After adjusting for inflation, determine by what percent the price of Brand A cereal went up or down from 1990 to 2000.

Answers to Practice Problems for Section 0.1

1. (a) Since the ratio of cola to lemon-lime soda is 4 to 3, for every 4 cups of cola, you must use 3 cups of lemon-lime soda. To make the mixture with 48 cups cola you'll need 12 groups of 4 cups of cola since $48 \div 4 = 12$, so you'll also need 12 groups of 3 cups of lemon-lime soda, which is $12 \times 3 = 36$ cups of lemon-lime soda. All together, this will make $12 \times (4+3) = 84$ cups of soda mixture. Table 13 summarizes this reasoning.

Another way to think about the problem is to view the soda mixture as made of 4 parts cola and 3 parts lemon-lime soda, as shown in the strip diagrams in Figure 9. Since you want the 4 parts of

cups cola:	4	$\xrightarrow{\times 12}$	48		4	$\xrightarrow{\times 12}$	48
cups lemon-lime:	3		?		3	$\xrightarrow{\times 12}$	36
total cups:	7		?		7	$\xrightarrow{\times 12}$	84

Table 13: Solving a Proportion by Using Reasoning about Multiplication and Division and a Ratio Table

cola to be 48 cups, each part must be $48 \div 4 = 12$ cups. Therefore the 3 parts lemon-lime soda are $3 \times 12 = 36$ cups. The full mixture consists of 7 parts, which is therefore $7 \times 12 = 84$ cups.

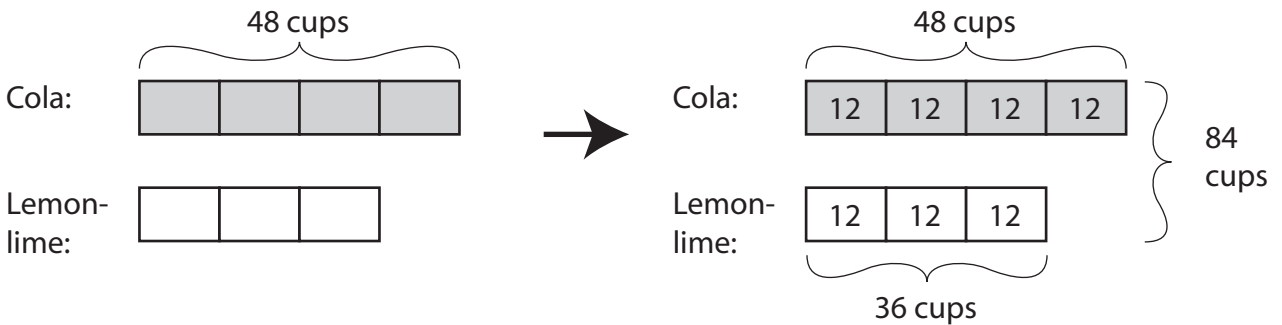


Figure 9: Solving a Proportion by Using Reasoning about Multiplication and Division and a Strip Diagram

- (b) Once again, for every 4 cups of cola, you must use 3 cups of lemon-lime soda. Combined, 4 cups of cola and 3 cups of lemon-lime soda make 7 cups of soda mixture. To make 105 cups of soda mixture you must use 15 groups of 7 cups of mixture since $105 \div 7 = 15$. Therefore you must also use 15 groups of 4 cups of cola, or $15 \times 4 = 60$ cups of cola and 15 groups of 3 cups of lemon-lime soda, or $15 \times 3 = 45$ cups of lemon-lime soda. Table 14 summarizes this reasoning.

Another way to think about the problem is to view the soda mixture as made of a total of 7 parts, of which 4 parts are cola and 3 parts are lemon-lime soda, as indicated in the strip diagrams in Figure 10. Since you want the 7 parts of soda mixture to be 105

cups cola:	4	?	4	$\xrightarrow{\times 15}$	60	
cups lemon-lime:	3	?	3	$\xrightarrow{\times 15}$	45	
total cups:	7	$\xrightarrow{\times 15}$	105	7	$\xrightarrow{\times 15}$	105

Table 14: Solving a Proportion by Using Reasoning about Multiplication and Division and a Ratio Table

cups, each part must be $105 \div 7 = 15$ cups. Therefore the 4 parts of cola are $4 \times 15 = 60$ cups and the 3 parts of lemon-lime soda are $3 \times 15 = 45$ cups.

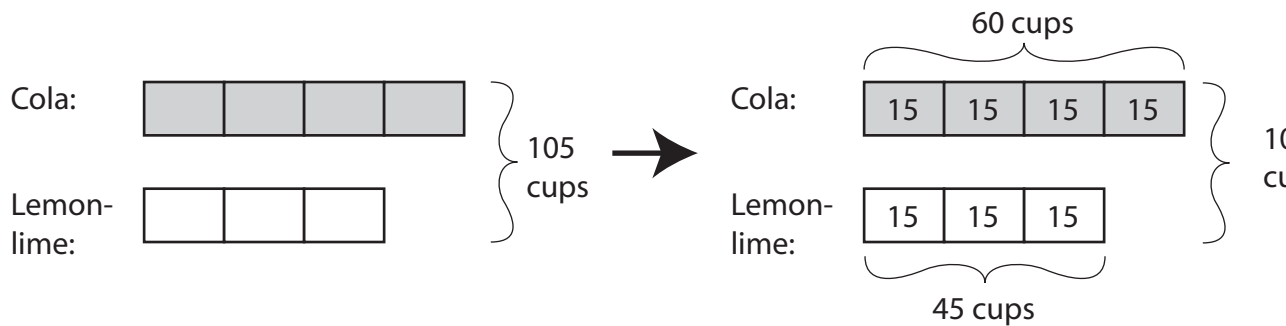


Figure 10: Solving a Proportion by Using Reasoning about Multiplication and Division and a Strip Diagram

2. (a) Since the race car goes 15 miles every 6 minutes, it goes 30 miles in 12 minutes and 45 miles in 18 minutes as shown in the double number line in Figure 11. Fifteen minutes is half way between 12 and 18 minutes, so the race car must go half way between 30 and 45 miles in 15 minutes. Since $45 - 30 = 15$ and $15 \div 2 = 7.5$, the race car goes $30 + 7.5 = 37.5$ miles in 15 minutes.
- (b) Since the race car goes 15 miles every 6 minutes, it goes 30 miles in 12 minutes, and 45 miles in 18 minutes, as shown in the double number line in Figure 12. Twenty miles is $\frac{1}{3}$ of the way between 15 miles and 30 miles. So it must take the race car the time that is $\frac{1}{3}$ of the way between 6 and 12 minutes to go 20 miles. Since

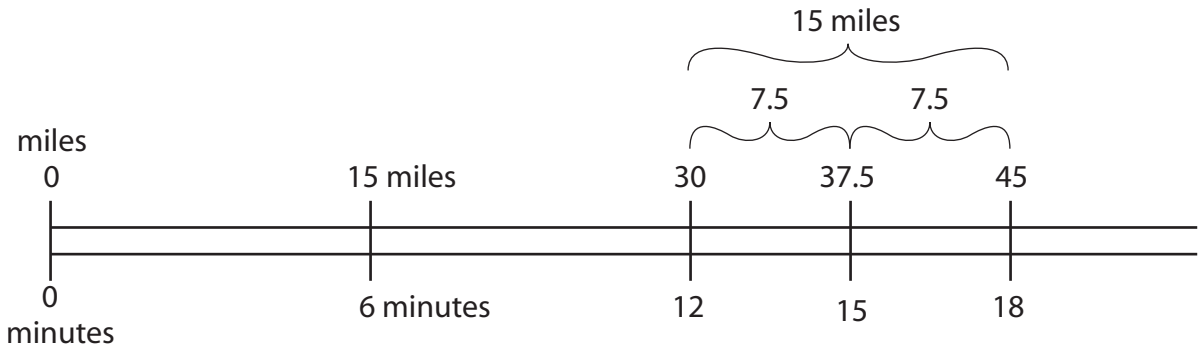


Figure 11: Using a Double Number Line to Solve a Proportion

$12 - 6 = 6$ and $6 \div 3 = 2$, one third of the way between 6 and 12 minutes is $6 + 2 = 8$ minutes.

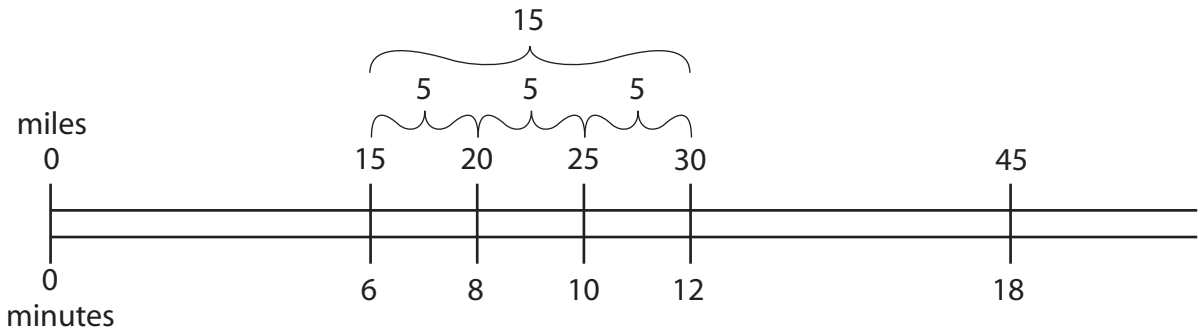


Figure 12: Using a Double Number Line to Solve a Proportion

3. Since the ratio of Quint's CDs to Chris's CDs is 7 to 3, Quint's CDs can be divided into 7 parts and Chris's CDs can be divided into 3 parts, where all parts are the same size, as shown in the strip diagram in Figure 13. Since the boys have the same number of CDs in the end, and since Quint has 4 more parts than Chris at first, two of Quint's parts must go to Chris. Those two parts are 6 CDs, so each part consists of 3 CDs. Therefore Quint had $7 \times 3 = 21$ CDs to start with and Chris had $3 \times 3 = 9$ CDs to start with.
4. The $3\frac{1}{2}$ cups of red paint and $4\frac{1}{2}$ cups of yellow paint combine to make $3\frac{1}{2} + 4\frac{1}{2} = 8$ cups orange paint. Since 12 cups is $1\frac{1}{2}$ times as much as

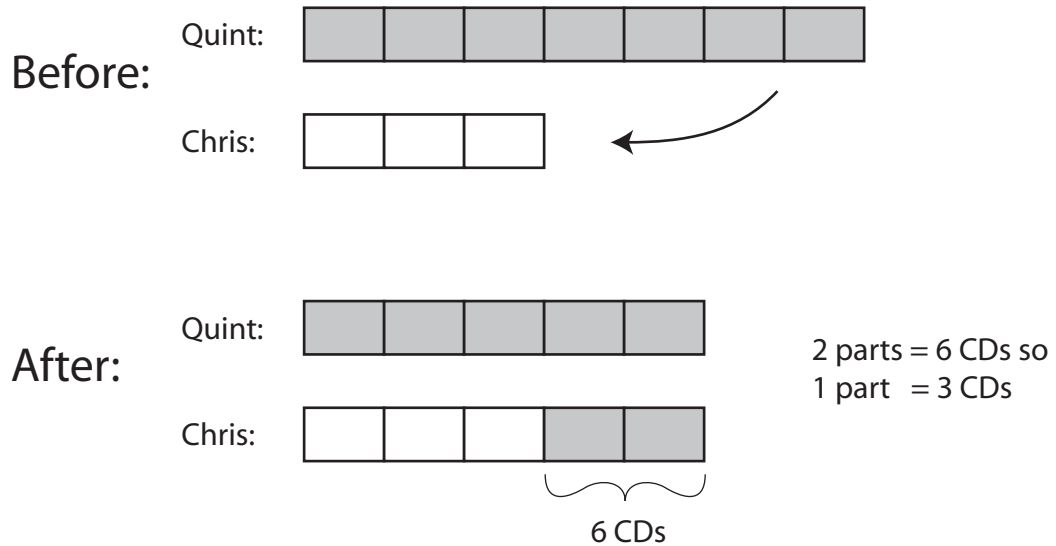


Figure 13: Solving a Ratio Problem By Moving Parts in a Strip Diagram

8 cups, Cali will need $1\frac{1}{2}$ times as much red paint and yellow paint. Therefore, Cali will need

$$1\frac{1}{2} \cdot 3\frac{1}{2} = \frac{3}{2} \cdot \frac{7}{2} = \frac{21}{4} = 5\frac{1}{4}$$

cups of red paint and

$$1\frac{1}{2} \cdot 4\frac{1}{2} = \frac{3}{2} \cdot \frac{9}{2} = \frac{27}{4} = 6\frac{3}{4}$$

cups of yellow paint. This reasoning is summarized in Table 15.

cups red:	$3\frac{1}{2}$?	$3\frac{1}{2}$	$\xrightarrow{\times 1\frac{1}{2}}$	$5\frac{1}{4}$	
cups yellow:	$4\frac{1}{2}$?	$4\frac{1}{2}$	$\xrightarrow{\times 1\frac{1}{2}}$	$6\frac{3}{4}$	
total cups orange:	8	$\xrightarrow{\times 1\frac{1}{2}}$	12	8	$\xrightarrow{\times 1\frac{1}{2}}$	12

Table 15: Solving a Proportion by Using Reasoning about Multiplication and Division and a Ratio Table

5. You can reason that $1\frac{1}{2}$ cups is 6 times as much as $\frac{1}{4}$ cup, so you use 6 times as much bubbly water as grape juice concentrate. To find out how much concentrate to use for 2 cups bubbly water we ask “6 times what is equal to 2 cups?”. The answer is $2 \div 6 = \frac{2}{6} = \frac{1}{3}$ cup, so you should use $\frac{1}{3}$ of a cup of grape juice concentrate for 2 cups bubbly water. This reasoning is summarized in Table 16.

cups bubbly water:	$1\frac{1}{2}$	2	$1\frac{1}{2}$	2
	$\uparrow \times 6$		$\uparrow \times 6$	$\uparrow \times 6$
cups concentrate:	$\frac{1}{4}$?	$\frac{1}{4}$	$\frac{1}{3}$

Table 16: Using Reasoning About Multiplication and Division to Solve a Proportion

Another way to reason is to “go through 1” cup of bubbly water by first going through a whole number of cups of bubbly water. If you use twice as much bubbly water, which is 3 cups, you should also use twice as much concentrate, which is $\frac{1}{2}$ of a cup. So, if you wanted to use 1 cup of bubbly water, which is $\frac{1}{3}$ as much as 3 cups, then you should use $\frac{1}{3}$ of $\frac{1}{2}$ cup of grape juice concentrate, namely $\frac{1}{6}$ of a cup. Then to use 2 cups of bubbly water, which is twice as much as 1 cup, you should use twice as much as $\frac{1}{6}$ of a cup of grape juice concentrate, namely, $\frac{2}{6} = \frac{1}{3}$ cups. This reasoning is summarized in Table 17.

cups bubbly water:	$1\frac{1}{2}$	$\xrightarrow{\times 2}$	3	$\xrightarrow{\div 3}$	1	$\xrightarrow{\times 2}$	2
cups concentrate:	$\frac{1}{4}$	$\xrightarrow{\times 2}$	$\frac{1}{2}$	$\xrightarrow{\div 3}$	$\frac{1}{6}$	$\xrightarrow{\times 2}$	$\frac{1}{3}$

Table 17: Using Reasoning About Multiplication and Division to Solve a Proportion, “Going Through 1” Method

6. “Direct” Method: Since $2\frac{1}{2}$ mL is $\frac{1}{4}$ of 10 mL, the pharmacist will also need $\frac{1}{4}$ as much water, which is $\frac{1}{4} \cdot 12 = 3$ mL of water. The medicine and water combined make $2\frac{1}{2} + 3 = 5\frac{1}{2}$ mL of medicine/water mixture. This reasoning is summarized in Table 18.

mL medicine:	10	$\xrightarrow{\div 4}$	$2\frac{1}{2}$
mL water:	12	$\xrightarrow{\div 4}$	3
mL total:	22	$\xrightarrow{\div 4}$	$5\frac{1}{2}$

Table 18: Solving a Proportion by Using Reasoning about Multiplication and Division and a Ratio Table, “Direct Method”

“Going through 1” Method: To make the mixture with 1 mL of medicine, which is $\frac{1}{10}$ as much as 10 mL of medicine, the pharmacist will also need $\frac{1}{10}$ as much water, namely $\frac{1}{10} \cdot 12 = \frac{6}{5} = 1\frac{1}{5}$ mL of water, thus making $1 + 1\frac{1}{5} = 2\frac{1}{5}$ mL of medicine/water mixture. Then to make the mixture with $2\frac{1}{2}$ mL of medicine, which is $2\frac{1}{2}$ times as much as 1 mL, the pharmacist must $2\frac{1}{2}$ times as much water, namely $2\frac{1}{2} \cdot 1\frac{1}{5} = \frac{5}{2} \cdot \frac{6}{5} = 3$ mL of water, thus making $5\frac{1}{2}$ mL of medicine/water mixture. This reasoning is summarized in Table 19.

mL medicine:	10	$\xrightarrow{\div 10}$	1	$\xrightarrow{\times 2\frac{1}{2}}$	$2\frac{1}{2}$
mL water:	12	$\xrightarrow{\div 10}$	$1\frac{1}{5}$	$\xrightarrow{\times 2\frac{1}{2}}$	3
mL total:	22	$\xrightarrow{\div 10}$	$2\frac{1}{5}$	$\xrightarrow{\times 2\frac{1}{2}}$	$5\frac{1}{2}$

Table 19: Solving a Proportion by Using Reasoning about Multiplication and Division and a Ratio Table, “Direct Method”

7. Out of the total 7 cups of purple paint, 3 cups are blue, so $\frac{3}{7}$ of the paint mixture is blue. Thinking in terms of division, if we imagine the blue paint divided equally among the 7 cups of purple paint, then there is $3 \div 7 = \frac{3}{7}$ cups of blue paint in each cup of purple paint. Similarly, 4 out of 7 cups of the purple paint are red, so $\frac{4}{7}$ of the paint is red. Imagining the red paint divided equally among the 7 cups of purple paint, there is $4 \div 7 = \frac{4}{7}$ cups of red paint in each cup of purple paint. Since there are 3 cups of blue paint and 4 cups of red paint in the

mixture, if we think of dividing the blue paint equally among the 4 cups of red paint, then there are $3 \div 4 = \frac{3}{4}$ cups blue paint for each cup of red paint. By the same logic, there are $4 \div 3 = \frac{4}{3} = 1\frac{1}{3}$ cups red paint for each cup of blue paint.

If we think of dividing the 7 cups of purple paint equally among the 3 cups of blue paint, then there are $7 \div 3 = \frac{7}{3} = 2\frac{1}{3}$ cups of purple paint for each cup of

blue paint. By the same logic, there are $7 \div 4 = \frac{7}{4} = 1\frac{3}{4}$ cups of purple paint for each cup of red paint.

8. Method 1: If we think of the 3 tablespoons of salt in the first mixture as being divided equally among the 4 cups of water, then each cup of water contains $3 \div 4 = \frac{3}{4}$ tablespoons of salt. Similarly, each cup of water in the second mixture contains $4 \div 5 = \frac{4}{5}$ tablespoons of salt. Since $\frac{4}{5} = 0.8$ and $\frac{3}{4} = 0.75$, and since $0.8 > 0.75$, the second mixture contains more salt per cup of water. Thus, it is more salty.

Method 2: If we make 5 batches of the first mixture and 4 batches of the second mixture, then both will contain 20 cups of water. The first mixture will contain $5 \times 3 = 15$ tablespoons of salt and the second mixture will contain $4 \times 4 = 16$ tablespoons of salt. Since both mixtures contain the same amount of water but the second mixture contains 1 more tablespoon of salt than the first, the second mixture must be more salty.

9. (a) Because the crew cuts 5 acres every 2 days, it will cut half as much in 1 day, namely $2\frac{1}{2}$ acres. To determine how many days it will take to cut 8 acres, we must determine how many groups of $2\frac{1}{2}$ are in 8, which is $8 \div 2\frac{1}{2} = 3\frac{1}{5}$ days.
- (b) In part (a) we saw that one crew cuts $2\frac{1}{2}$ acres per day. Therefore, three crews will cut 3 times as much per day, which is $3 \cdot 2\frac{1}{2} = 7\frac{1}{2}$ acres per day. To determine how many days it will take to cut 10 acres, we must figure out how many groups of $7\frac{1}{2}$ are in 10. This is solved by the division problem $10 \div 7\frac{1}{2}$. Because $10 \div 7\frac{1}{2} = 10 \div \frac{15}{2} = \frac{10}{1} \cdot \frac{2}{15} = \frac{20}{15} = 1\frac{5}{15} = 1\frac{1}{3}$, we conclude that it will take the 3 crews $1\frac{1}{3}$ days to cut 10 acres.

10. The proportion

$$\frac{3 \text{ people}}{2 \text{ days}} = \frac{2 \text{ people}}{x \text{ days}}$$

is not valid for this situation because, when more people are painting, it will take less time to paint a fence. It is not the case that, for each group of 3 people there are, it will take 2 days to paint a fence. Therefore, the relationship between the number of people and the number of days it takes to paint a fence is not a ratio.

Thinking logically, if 3 people take 2 days to paint 5 fences, then those 3 people will take $2 \div 5 = \frac{2}{5}$ of a day to paint just one fence (dividing the 2 days equally among the 5 fences). If just one person was painting, it would take 3 times as long to paint the fence, namely, $3 \cdot \frac{2}{5} = \frac{6}{5}$ of a day (which is $1\frac{1}{5}$ days). With 2 people painting, it will take half as much time to paint the fence, namely, $\frac{3}{5}$ of a day.

11. The CPI was 29.6 in 1960 and 172.2 in 2000. Therefore, \$.05 in 1960 had the same buying power as did

$$\frac{172.2}{29.6} \cdot \$.05 = \$.29$$

in 2000.

12. (a) In 2000, the CPI was 172.2, so \$50,000 could buy $50,000 \div 172.2 = 290.36$ “units” of a market basket of goods and services. In 1950 the CPI was 24.1, so the same 290.36 units of a market basket of goods and services cost $290.36 \times \$24.10 = \6998 in 1950. Therefore, \$6998 in 1950 had the same buying power as did \$50,000 in 2000.
- (b) In 2000, the CPI was 172.2; in 1980, the CPI was 82.4. Thus, a “unit” of a market basket of goods and services cost $82.4 \div 172.2 = .47851$ times as much in 1980 as it did in 2000. Therefore, \$50,000 in 2000 had the same buying power as did $.47851 \times \$50,000 = \$23,926$ in 1980.

13. According to the CPI, \$5.00 in 1990 had the same buying power as did

$$\frac{172.2}{130.7} \cdot \$5.00 = \$6.59$$

in 2000. So if an item cost \$5.00 in 1990 and \$6.59 in 2000, then its price went up a little slower than inflation.

14. In 2000, the CPI was 172.2 and in 1990, the CPI was 130.7. Therefore, adjusting for inflation, a price of \$2.39 in 1990 corresponds to

$$\frac{172.2}{130.7} \cdot \$2.39 = \$3.15$$

in 2000. Thus, to make a fair comparison between the prices, we should calculate the percent increase from \$3.15 to \$3.89, instead of the percent increase from \$2.39 to \$3.89. Because

$$\frac{3.89}{3.15} = 1.23$$

the price of the cereal increased by 23%, after adjusting for inflation.

Problems for Section 0.1

- Without using a calculator, explain how you can use reasoning and mental arithmetic to determine which of the following two laundry detergents is a better buy (i.e., has a lower rate of dollars per load washed):
 - a box of laundry detergent that washes 80 loads and costs \$12.75
 - a box of laundry detergent that washes 36 loads and costs \$6.75
- You can make grape juice by mixing 1 can of frozen grape juice concentrate with 3 cans of water. Use simple reasoning with multiplication and division to find at least 4 other ways of mixing grape juice concentrate with water so that the result will be in the same ratio and therefore taste the same. At least 2 of your ways should involve numbers that are not whole numbers.
- You can make a pink paint by mixing $2\frac{1}{2}$ cups white paint with $1\frac{3}{4}$ cups red paint. Use simple reasoning with multiplication and division to find at least 4 other ways of mixing white paint and red paint in the same ratio in order to make the same shade of pink paint.
- (c) Walking at a constant speed, a person walks $\frac{3}{4}$ of a mile every 12 minutes. Use simple, logical reasoning supported by a double number line to help you determine the answers to the next questions.
 - How far does the person walk in 30 minutes?
 - How long does it take the person to walk $2\frac{1}{2}$ miles?
- In a terrarium, the ratio of grasshoppers to crickets is 6 : 5. There are 48 grasshoppers. How many crickets are there? Explain how to solve

this problem in two ways: with the aid of a strip diagram and with the aid of a ratio table. In each case, be sure to discuss the reasoning involved in the method, in other words, explain why the method makes sense.

6. (c) At a zoo, the ratio of King Penguins to Emperor Penguins is $2 : 3$. In all, there are 45 King and Emperor Penguins combined. How many Emperor Penguins are at the zoo? Explain how to solve this problem in two ways: with the aid of a strip diagram and with the aid of a ratio table. In each case, be sure to discuss the reasoning involved in the method, in other words, explain why the method makes sense.
7. On a farm, the ratio of grey goats to white goats is $2 : 5$. There are 100 grey goats. How many white goats are there? Explain how to solve this problem in two ways: with the aid of a strip diagram and with the aid of a ratio table. In each case, be sure to discuss the reasoning involved in the method, in other words, explain why the method makes sense.
8. (c) An orange-lemon juice mixture can be made by mixing orange juice and lemonade in a ratio of 5 to 2. Explain how to solve each of the following problems about the juice mixture by using logical reasoning about multiplication and division in two ways: with the aid of a ratio table and with the aid of a strip diagram.
 - (a) How much orange juice and how much lemonade should you use to make the juice mixture with 75 cups of orange juice? How much juice mixture will this make?
 - (b) How much orange juice and how much lemonade should you use to make 140 cups of the juice mixture?
 - (c) How much orange juice and how much lemonade should you use to make the juice mixture with 15 cups of lemonade? How much juice mixture will this make?
 - (d) How much orange juice and how much lemonade should you use to make 10 cups of the juice mixture?
9. (This problem refers to problem 8 (a).) Create two new problems for your students by changing the ratio 5 to 2 and the number, 75, of cups of juice mixture in problem 8 (a). One problem should be about the

same level of difficulty as problem 8 (a) and the other problem should be harder. Say which problem is which and why. Explain how to solve each problem in two different ways.

10. Brad made some punch by mixing $\frac{1}{2}$ of a cup of grape juice with $\frac{1}{4}$ cup of sparkling water. Brad really likes his punch, so he decides to make a larger batch of it using the same ratio.
 - (a) Brad wants to make 6 cups of his punch. How much grape juice and how much sparkling water should Brad use? Explain how to use simple reasoning to solve this problem in two different ways.
 - (b) Now Brad wants to make 4 cups of his punch. How much grape juice and how much sparkling water should Brad use? Explain how to use simple reasoning to solve this problem in two different ways.
11. To make grape juice using frozen juice concentrate you must mix the frozen juice concentrate with water in a ratio of 1 to 3. How much frozen juice concentrate and how much water should you use to make $1\frac{1}{2}$ cups of grape juice? Solve this problem in two different ways, explaining your reasoning in each case.
12. You can make a soap bubble mixture by combining 2 tablespoons water with 1 tablespoon liquid dishwashing soap and 4 drops of corn syrup. Using the same ratios, how much liquid dishwashing soap and how many drops of corn syrup should you use if you want to make soap bubble mixture using 5 tablespoons of water? Solve this problem using only simple reasoning about multiplication or division (or both). Explain your reasoning.
13. You can make concrete by mixing 1 part cement with 2 parts pea gravel and 3 parts sand. Using the same ratios, how much cement and how much pea gravel should you use if you want to make concrete with 8 cubic feet of sand? Solve this problem using only simple reasoning about multiplication or division (or both). Explain your reasoning.
14. (a) John was paid \$250 for $3\frac{3}{4}$ hours of work. At that rate, how much should John make for $2\frac{1}{2}$ hours of work? Use the most elementary reasoning you can to solve this problem. Explain your reasoning.

- (b) John was paid \$250 for $3\frac{3}{4}$ hours of work. At that rate, how long should John be willing to work for \$100? Use the most elementary reasoning you can to solve this problem. Explain your reasoning.
15. The ratio of Frank's marbles to Huang's marbles is 3 to 2. After Frank gives $\frac{1}{2}$ of his marbles to another friend, Frank has 30 fewer marbles than Huang. How many marbles does Huang have?
- (a) Explain how to solve the problem with the aid of a strip diagram.
- (b) Create an easier problem for your students by changing the ratio, 3 to 2, to a different ratio and by changing the number of marbles, 30, to a different number of marbles in the problem. Make sure the problem has a sensible answer. Explain how to solve the problem.
- (c) Create a problem of about the same level of difficulty as the original problem by changing the ratio, 3 to 2, to a different ratio and by changing the number of marbles, 30, to a different number of marbles in the problem. Make sure the problem has a sensible answer. Explain how to solve the problem.
16. Asia and Taryn each had the same amount of money. After Asia spent \$14 and Taryn spent \$22, the ratio of Asia's money to Taryn's money was 4 to 3. How much money did each girl have at first?
- (a) Explain how to solve the problem with the aid of a strip diagram.
- (b) Create a harder problem for your students by changing the ratio, 4 to 3, to a different ratio and by changing the dollar amounts, \$14 and \$22 to different dollar amounts in the problem. Explain how to solve the problem.
- (c) Create a problem of about the same level of difficulty as the original problem by changing the ratio, 4 to 3, to a different ratio and by changing the dollar amounts, \$14 and \$22 to different dollar amounts in the problem. Explain how to solve the problem.
17. An aquarium contained an equal number of horseshoe crabs and sea stars. After 15 horseshoe crabs were removed and 27 sea stars were removed, the ratio of horseshoe crabs to sea stars was 5 : 3. Explain how to solve this problem in two ways: with the aid of a strip diagram and by setting up and solving a proportion.

18. Pat mixed 2 cups of blue paint with 5 cups of yellow paint to make a green paint. For each of the following fractions and division problems, interpret the fraction or the division problem in terms of Pat's paint mixture and explain why your interpretation makes sense. Use the definition of fraction from Chapter 3, do not simply refer to the fractions as ratios.

$$\frac{2}{5} \text{ or } 2 \div 5; \quad \frac{5}{2} \text{ or } 5 \div 2; \quad \frac{2}{7} \text{ or } 2 \div 7;$$

$$\frac{7}{2} \text{ or } 7 \div 2; \quad \frac{5}{7} \text{ or } 5 \div 7; \quad \frac{7}{5} \text{ or } 7 \div 5$$

19. (a) Which of the following two mixtures will have a stronger lime flavor?

- 2 parts lime juice concentrate mixed in 5 parts water
- 4 parts lime juice concentrate mixed in 7 parts water

Solve this problem in two different ways, explaining in detail why you can solve the problem the way you do. In particular, if you use fractions in your explanation, be sure to explain how the fractions are relevant.

- (b) A student might say that the second mixture has a stronger flavor than the first mixture because the numbers for the second mixture are greater (in other words, $4 > 2$ and $7 > 5$). Even if the conclusion is correct, is the student's reasoning valid? Explain why or why not.
20. (a) Snail A moved 6 feet in 7 hours. Snail B moved 7 feet in 8 hours. Both snails moved at constant speeds. Which snail went faster? Solve this problem in two different ways, explaining in detail why you can solve the problem the way you do. In particular, if you use fractions in your explanation, be sure to explain how the fractions are relevant.
- (b) A student might say that snail B moved faster than snail A because the numbers for snail B are greater (in other words, $7 > 6$ and $8 > 7$). Even if the conclusion is correct, is the student's reasoning valid? Explain why or why not.
21. (c) Allie, Benton, and Cathy are planning to mix red and yellow paint. They are considering which of the two following paint mixtures will make a more yellow paint:

- a mixture of 3 parts red to 5 parts yellow
- a mixture of 4 parts red to 6 parts yellow

Allie says that both paints will look the same because to make the second mixture you just add one part of each color to the first mixture. Because you add the same amount of each color, the second mixture should look the same as the first mixture. Benton says that the second mixture should be more yellow than the first because it uses more yellow than the first mixture. Cathy says that both paints should look the same because each uses 2 parts more yellow than red.

- Discuss the children's ideas. Is their reasoning valid or not?
 - Which paint will be more yellow and why? Solve this problem in two different ways, explaining in detail why you can solve the problem the way you do. In particular, if you use fractions in your explanation, be sure to explain how the fractions are relevant.
22. (c) A dough recipe calls for 3 cups of flour and $1\frac{1}{4}$ cups of water. You want to use the same ratio of flour to water to make a dough with 10 cups of flour. How much water should you use?
- Solve this problem by setting up a proportion in which you set two fractions equal to each other.
 - Interpret the two fractions that you set equal to each other in part (a) in terms of the recipe. Explain why it makes sense to set these two fractions equal to each other.
 - Why does it make sense to cross-multiply the two fractions in part (a)? What is the logic behind the procedure of cross-multiplying?
 - Now solve the problem of how much water to use for 10 cups of flour in a different way, by using the most elementary reasoning you can. Explain your reasoning clearly.
23. A recipe that serves 6 people calls for $1\frac{1}{2}$ cups of rice. How much rice will you need to serve 8 people (assuming that the ratio of people to cups of rice stays the same)?
- Solve this problem by setting up a proportion in which you set two fractions equal to each other.

- (b) Interpret the two fractions that you set equal to each other in part (a) in terms of the recipe. Explain why it makes sense to set these two fractions equal to each other.
 - (c) Why does it make sense to cross-multiply the two fractions in part (a)? What is the logic behind the procedure of cross-multiplying?
 - (d) Now solve the same problem in a different way by using logical thinking and by using the most elementary reasoning you can. Explain your reasoning clearly.
24. Marge made light blue paint by mixing $2\frac{1}{2}$ cups of blue paint with $1\frac{3}{4}$ cups of white paint. Homer poured another cup of white paint in Marge's paint mixture. How many cups of blue paint should Marge add to bring the paint back to its original shade of light blue (mixed in the same ratio as before)? Use the most elementary reasoning you can to solve this problem. Explain your reasoning.
25. A batch of lotion was made at a factory by mixing 1.3 liters of ingredient A with 2.7 liters of ingredient B in a mixing vat. By accident, a worker added an extra 0.5 liters of ingredient A to the mixing vat. How many liters of ingredient B should the worker add to the mixing vat so that the ingredients will be in the original ratio? Use the most elementary reasoning you can to solve this problem. Explain your reasoning.
26. If a $\frac{3}{4}$ cup serving of snack food gives you 60% of your daily value of calcium, then what percent of your daily value of calcium is in $\frac{1}{2}$ cup of the snack food?
- (a) Solve the problem with the aid of a picture. Explain how your picture helps you to solve the problem.
 - (b) Now solve the problem numerically. Show your work.
27. If \$6,000 is 75% of a company's budget for a project, then what percent of the budget is \$10,000?
- (a) Solve the problem with the aid of a picture. Explain how your picture helps you to solve the problem.
 - (b) Now solve the problem numerically. Show your work.

28. Amy mixed 2 tablespoons of chocolate syrup in $\frac{3}{4}$ cup of milk to make chocolate milk. To make chocolate milk that is mixed in the same ratio and therefore tastes the same as Amy's, how much chocolate syrup will you need for 1 gallon of milk? Express your answer using appropriate units. Explain your reasoning. Recall that: 1 gallon = 4 quarts, 1 quart = 2 pints, 1 pint = 2 cups, 1 cup = 8 fluid ounces, 1 fluid ounce = 2 tablespoons.
29. A 5 gallon bucket filled with water is being pulled from the ground up to a height of 20 feet at a rate of 2 feet every 15 seconds. The bucket has a hole in it, so that water leaks out of the bucket at a rate of 1 quart ($\frac{1}{4}$ gallon) every 3 minutes. How much of the water will be left in the bucket by the time the bucket gets to the top? Solve this problem by using logical thinking and by using the most elementary reasoning you can. Explain your reasoning clearly.
30. Suppose that you have two square garden plots: one is 10 feet by 10 feet and the other is 15 feet by 15 feet. You want to cover both gardens with a one-inch layer of mulch. If the 10-by-10 garden took $3\frac{1}{2}$ bags of mulch, could you calculate how many bags of mulch you'd need for the 15-by-15 garden by setting up the proportion

$$\frac{3\frac{1}{2}}{10} = \frac{x}{15}?$$

Explain clearly why or why not. If the answer is no, is there another proportion that you could set up? It may help you to draw pictures of the gardens.

31. If you can rent 5 DVDs for 5 nights for \$5, then at that rate, how much should you expect to pay to rent 1 DVD for 1 night? Solve this problem by using logical thinking and the most elementary reasoning you can. Explain your reasoning clearly.
32. If a crew of 3 people take $2\frac{1}{2}$ hours to clean a house, then how long should a crew of 2 take to clean the same house? Assume that all people in the cleaning crew work at the same steady rate. Solve this problem by using logical thinking and the most elementary reasoning you can. Explain your reasoning clearly.

33. (c) If 6 men take 3 days to dig 8 ditches, then how long would it take 4 men to dig 10 ditches? Assume that all the ditches are the same size and take equally long to dig, and that all the men work at the same steady rate. Solve this problem by using logical thinking and the most elementary reasoning you can. Explain your reasoning clearly.
34. If liquid pouring at a steady rate from hose A takes 15 minutes to fill a vat and liquid pouring at a steady rate from hose B takes 10 minutes to fill the same vat, then how long will it take for liquid pouring from both hose A and hose B to fill the vat? Solve this problem by using logical thinking and the most elementary reasoning you can. Explain your reasoning clearly.
35. Suppose that 400 pounds of freshly picked tomatoes are 99% water by weight. After one day, the same tomatoes only weigh 200 pounds due to evaporation of water. (The tomatoes consist of water and solids. Only the water evaporates; the solids remain.)
- (a) How many pounds of solids are present in the tomatoes? (Notice that this is the same when they are freshly picked as after one day.)
 - (b) Therefore, when the tomatoes weigh 200 pounds, what percent of the tomatoes is water?
 - (c) Is it valid to use the following proportion to solve for the percent of water, x , in the tomatoes when they weigh 200 pounds?

$$\frac{0.99}{400} = \frac{x}{200}$$

If not, why not?

36. (+) If a loaf of bread cost \$3 in 2002, then how much should a loaf of bread have cost in 1960 based on the CPI? Explain your answer.
37. (+)
- (a) What amount of money had the same buying power in 1970 as \$30,000 did in 2000? Explain your answer.
 - (b) What amount of money had the same buying power in 1990 as \$30,000 did in 2000? Explain your answer.

38. (+) If an item cost \$3.50 in 1980 and \$4.50 in 2000, did its price go up faster, slower or the same as inflation? Explain your answer.
39. (+) Suppose that from 1995 to 2000 the price of a box of laundry detergent went from \$4.85 to \$5.95. After adjusting for inflation, determine by what percent the price of the box of laundry detergent went up or down from 1995 to 2000. Explain your answer.
40. (+) Josie made \$40,000 in 1995 and \$50,000 in 2000. In absolute dollars, that's a 25% increase in Josie's salary from 1995 to 2000. But what is the difference in buying power? After adjusting for inflation, how much higher or lower was Josie's salary in 2000 than it was in 1995? Give your answer as a percent. Explain your reasoning.
41. (+) Suppose that a person earned \$30,000 in 1995 and got a 3% raise every year after that until the year 2000 (so that every year, her salary was 3% higher than her previous year's salary). After adjusting for inflation, how much higher or lower was the person's salary in 2000 than it was in 1995? Give your answer as a percent. Explain your reasoning.
42. (+) Huffington University spent 31.1% of its total budget on instruction in 1990 and 18.7% of its total budget on instruction in 2000. In 1990, Huffington University spent \$42,672,636 on instruction, whereas in 2000, Huffington University spent \$139,672,851 on instruction. The total enrollment at Huffington University in 1990 was 22,879, and in 2000 it was 29,404. How can we compare expenditures on instruction in 1990 and 2000? We can take several different approaches, as in the following examples:
- John says that expenditures on instruction went down by 12.4%.
- Alice says that expenditures on instruction went down by 39.9%.
- Richard says that expenditures on instruction went up by 227%.
- Tia says that Richard's number is misleading because it doesn't adjust for inflation.
- Beatrice says that the number of students should also be taken into account.

(a) In what sense is John correct?

- (b) In what sense is Alice correct? (Compare the percentages—treat them as ordinary numbers, such as the prices of shoes. If a pair of shoes goes from \$31.10 to \$18.70, what percent decrease does that represent?)
- (c) In what sense is Richard correct?
- (d) Address Tia's objection. In 1990 the CPI was 130.7; in 2000 the CPI was 172.2. If you adjust for inflation and work in 2000 dollars, then by what percent did expenditures on instruction go up or down from 1990 to 2000?
- (e) Address Beatrice's point. Compare inflation adjusted expenditures on instruction per student in 1990 and 2000. By what percent did the per student expenditures on instruction go up or down?

