Sample Problems for MATH 5001, University of Georgia

1. Give three different decimals that the bundled toothpicks in Figure 1 could represent. In each case, explain why the bundled toothpicks can represent that decimal.

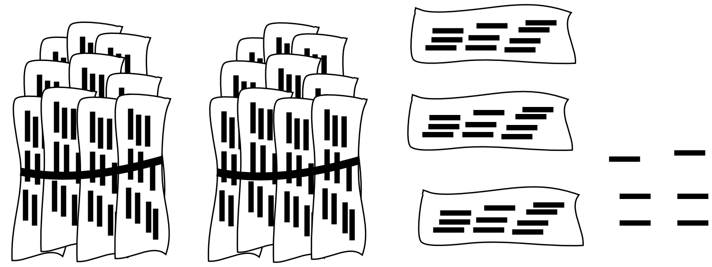


Figure 1: Which Decimals Can These Bundled Toothpicks Represent?

2. Label the tick marks on the three number lines in Figure 2 in three different ways. In each case, your labeling should fit with the fact that the tick marks at the ends of the number lines are longer than the other tick marks. You may further lengthen the tick marks at either end as needed.

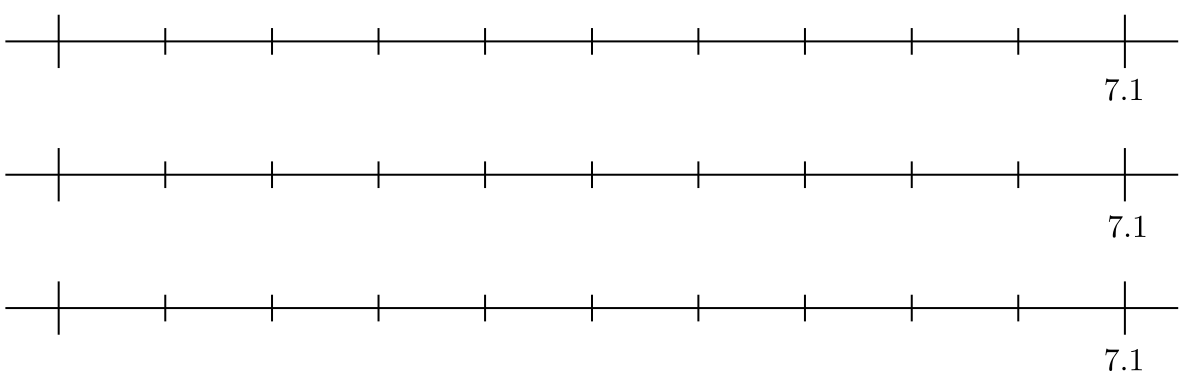


Figure 2: Label These Number Lines

3. Anna says that the dark blocks pictured in Figure 3 can't represent $\frac{1}{4}$ because there are 6 dark blocks and 6 is more than 1 but $\frac{1}{4}$ is supposed to be less than 1. What must Anna learn about fractions in order to overcome her confusion?

4. (a) Give three different fractions that you can legitimately use to describe the shaded region in Figure 4. For each fraction, explain why you can use that fraction to describe the shaded region.

   (b) Write an unambiguous question about the shaded region in Figure 4 that can be answered by naming a fraction. Explain why your question is not ambiguous.

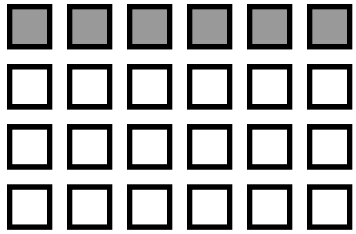


Figure 3: Representing the Fraction $\frac{1}{4}$

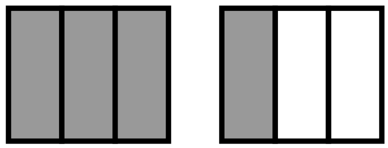


Figure 4: What Fraction is Shaded?

5. If $\frac{3}{4}$ of a cup of a food gives you your daily value of potassium, then what fraction of your daily value of potassium is in 1 cup of the food? Draw a picture that helps you solve this problem. Use your picture to help you explain your solution. For each fraction in this problem, and in your solution, describe the *whole* associated with this fraction. In other words, describe what each fraction is *of*.

6. Use the *meaning of fractions* to explain clearly why

$$\frac{2}{3} = \frac{2 \cdot 4}{3 \cdot 4}.$$

(Do not use multiplication by 1 to explain this.)

7. Plot $\frac{5}{6}$, $\frac{5}{4}$, and $\frac{4}{3}$ on the number line in Figure 5 in such a way that each number falls on a tick mark. Lengthen the tick marks of whole numbers.

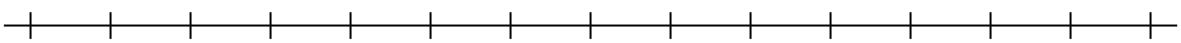


Figure 5: A Number Line

8. If the normal rainfall for August is 2.5 inches, but only 1.75 inches of rain fell in August, then what percent of the normal rainfall fell in August?
   (a) Show how to solve the problem with the aid of a picture. Explain how your picture helps you solve the problem.
   (b) Explain how to solve the problem numerically.

9. Find a number between 7.8651 and 7.8652 and plot all three numbers visibly and distinctly on a copy of the number line in Figure 6. Label all the longer tick marks on your number line.

10. Which of the following could the pictures in Figure 7 be used to illustrate? Circle all that apply.

$120 > 45$ $\quad$ $12 > 4.5$ $\quad$ $12 > .45$ $\quad$ $.12 > .45$ $\quad$ $.12 > .045$

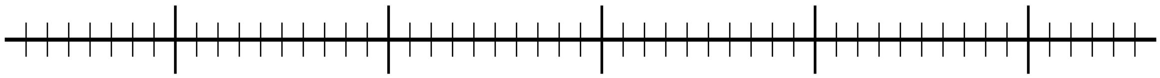


Figure 6: A Number Line

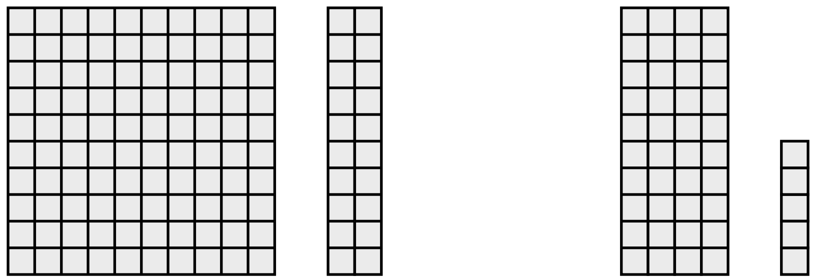


Figure 7: Which Inequalities Can These Blocks Represent?

11. Explain clearly and in detail why we can determine which of two fractions is greater by using the *cross-multiplying* method. What is the rationale behind this method? What are we really doing when we cross-multiply in order to compare fractions?

12. Conrad says that $\frac{3}{8} > \frac{2}{7}$ because $3 > 2$ and $8 > 7$. Regardless of whether or not Conrad's conclusion is correct, discuss whether or not Conrad's *reasoning* is valid.

13. Use reasoning *other than* converting to decimals, using common denominators, or cross-multiplying to determine which of $\frac{38}{39}$ and $\frac{45}{46}$ is greater. Explain your reasoning clearly and in detail.

14. Maya has made up her own method of rounding. Starting at the right-most place in a decimal, she keeps rounding to the value of the next place to the left until she reaches the place to which the decimal was to be rounded. For example, Maya would use the following steps to round 3.2716 to the nearest tenth:

    $$3.2716 \to 3.272 \to 3.27 \to 3.3.$$

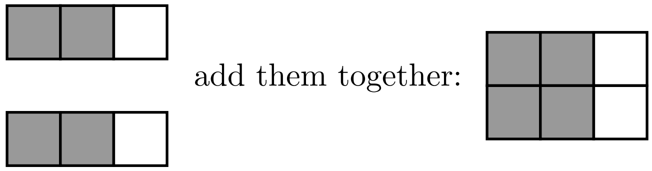
    Is Maya's method a valid way to round? Explain why or why not.

15. Erin wants to figure out how much time elapsed from 10:55 am to 11:30 am. Erin does the following:

    $$\begin{array}{r} {\scriptstyle 0\ \ 12\,10} \\ 1\,\not{1}:\not{3}\,\not{0} \\ -10:55 \\ \hline 0:75 \end{array}$$

    and says the answer is 75 minutes. Is Erin right? If not, explain what is wrong with her method and show how to *modify her method* to make it correct.

16. Frank says that $\frac{2}{3} + \frac{2}{3} = \frac{4}{6}$ and uses the picture below as evidence. Explain why Frank's method is not a valid way to add fractions. Be specific. (*Do not* explain how to do the problem correctly, explain where the flaw is in Frank's reasoning.)



17. Which of the following problems can be solved by adding $\frac{1}{3}+\frac{1}{4}$? For those problems that can't be solved by adding $\frac{1}{3}+\frac{1}{4}$, solve the problem in another way if there is enough information to do so, or explain why the problem cannot be solved.

    (a) One third of the boys in Mrs. Scott's class want to have a peanut butter sandwich for lunch. One fourth of the girls in Mrs. Scott's class want to have a peanut butter sandwich for lunch. What fraction of the children in Mrs. Scott's class want to have a peanut butter sandwich for lunch?

    (b) One third of the pizzas served at a party have pepperoni on them. One fourth of the pizzas served at the party have mushrooms on them. What fraction of the pizzas served at the party have either pepperoni or mushrooms on them?

    (c) The pizzas served at a party all have only one topping. One third of the pizzas served at the party have pepperoni on them. One fourth of the pizzas served at the party have mushrooms on them. What fraction of the pizzas served at the party have either pepperoni or mushrooms on them?

18. Can the following story problem be solved by subtracting $\frac{1}{3}-\frac{1}{4}$? If not, explain why not, and solve the problem in another way if there is enough information to do so.

    Story problem: There is $\frac{1}{3}$ of a pie left over from yesterday. Julie eats $\frac{1}{4}$ of the leftover pie. Now how much pie is left?

19. A community goes from producing $2\frac{1}{2}$ tons of waste per month to producing $3\frac{1}{2}$ tons of waste per month.

    (a) Show how to use a picture to help you calculate the percent increase in a community's monthly waste production.

    (b) Show how to calculate the percent increase in the community's monthly waste production numerically.

20. Tomaslav has learned the following facts well:

    - all the sums of whole numbers that add to 10 or less; Tomaslav knows these facts "forwards and backwards," for example, he knows not only that $5+2$ is 7, but also that 7 breaks down into $5+2$
    - $10+1, \quad 10+2, \quad 10+3, \ldots, 10+10$
    - the *doubles* $1+1, \quad 2+2, \quad 3+3, \ldots, 10+10$

    Describe three different ways that Tomaslav could use reasoning together with the facts he knows well to solve $8+7$. Draw pictures to support your descriptions. In each case, write equations to go along with the strategies you describe. Take care to use parentheses appropriately and as needed.

21. To solve $341 - 176$, a student writes the following:

$$\begin{array}{r} 341 \\ \underline{176} \\ -5 \\ -30 \\ 200 \end{array} \qquad \begin{array}{r} 200 \\ \underline{-30} \\ 170 \\ \underline{-5} \\ 165 \end{array} \qquad 341 - 176 = 165$$

Describe the student's solution strategy and discuss why the strategy makes sense. Expanded forms may be helpful to your discussion.

22. A rug is 4 feet wide and 5 feet long. Use the *meaning of multiplication* to explain why we can calculate the area of the rug by multiplying.

23. A box is 2 feet deep, 3 feet wide, and 4 feet tall. Use the *meaning of multiplication* to explain why we can calculate the volume of the box by multiplying.

24. Which property or properties of arithmetic do you use when you calculate 3×70 by first calculating $3 \times 7 = 21$ and then putting a zero on the end of 21 to make 210? Write equations to show which properties are used and where.

25. Write at least two different expressions for the total number of triangles in Figure 8. Each expression is *only* allowed to use the numbers 3, 4, and 5, the multiplication symbol, and parentheses. In each case, use the meaning of multiplication to explain why your expression represents the total number of triangles in Figure 8.

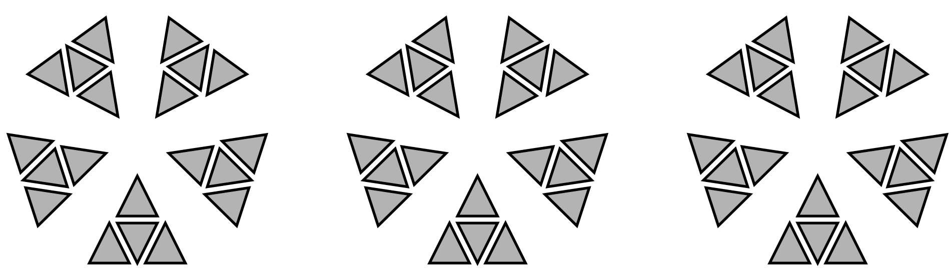


Figure 8: How Many Triangles?

26. Keisha says that it's easy to multiply even numbers by 5 because you just take half of the number and put a zero on the end. Write equations that incorporate Keisha's method and that demonstrate why her method is valid. Use the case 5×8 for the sake of concreteness. Write your equations in the following form:

$$\begin{aligned} 5 \times 8 &= \text{some expression} \\ &= \text{some expression} \\ &\vdots \quad \vdots \\ &= 40. \end{aligned}$$

27. Ashley knows her $1\times$, $2\times$, $3\times$, $4\times$, and $5\times$ multiplication tables well.

(a) Describe how the three pictures in Figure 9 provide Ashley with three different ways to determine 6×8 from multiplication facts that she already knows well. In each case, write an equation that corresponds to the picture and that shows how 6×8 is related to other multiplication facts.

Figure 9: Different Ways to Think of 6×8

(b) Draw pictures showing two different ways that Ashley could use the multiplication facts she knows well to determine 6×7. In each case, write an equation that corresponds to the picture and that shows how 6×7 is related to other multiplication facts.

28. Halley calculates 45% of 280 in the following way:

> Half of 280 is 140. I know 10% is 28, so 5% is half of that, which is 14.
> So I get 140 minus 14, which is 126.

(a) Explain briefly why it makes sense for Halley to solve the problem the way she does. What is the idea behind her strategy?

(b) Write a string of equations that incorporate Halley's ideas. Which properties of arithmetic did Halley use (knowingly or not) and where? Be thorough and be specific. Write your equations in the following format:

$$\begin{aligned} 45\% \times 280 &= \text{some expression} \\ &= \text{some expression} \\ &\vdots\quad \vdots \\ &= 126. \end{aligned}$$

29. (a) Use the *partial products* algorithm to calculate $\begin{array}{r} 34 \\ \times 27 \\ \hline \end{array}$

(b) Use the *meaning of multiplication* and a *picture* to give a clear and thorough explanation for why the partial products algorithm gives the correct answer to the multiplication problem in part (a). (Use graph paper for your picture.)

(c) Show why the partial products algorithm calculates the correct answer to the multiplication problem in part (a) by writing equations that use *properties of arithmetic* and that incorporate the calculations of the partial products algorithm. (FOIL is *not* a property of arithmetic.) Write your equations in the following format:

$$\begin{aligned} 34 \times 27 &= \text{some expression} \\ &= \text{some expression} \\ &\vdots \end{aligned}$$

Identify the properties of arithmetic that you used and show where you used them.

(d) Relate your equations for part (c) to your picture for part (b).

30. Which of the following are story problems for $\frac{1}{2} \times \frac{3}{4}$ and which are not? Explain briefly in each case.

    (a) There is $\frac{3}{4}$ of a cake left. One half of the children in Mrs. Brown's class want cake. How much of the cake will the children get?

    (b) A brownie recipe used $\frac{3}{4}$ of a cup of butter for a batch of brownies. You ate $\frac{1}{2}$ of a batch. How much butter did you consume when you ate those brownies?

    (c) Three quarters of a pan of brownies is left. Johnny eats $\frac{1}{2}$ of a pan of brownies. Now what fraction of a pan of brownies is left?

    (d) Three quarters of a pan of brownies is left. Johnny eats $\frac{1}{2}$ of what is left. How many brownies did Johnny eat?

    (e) Three quarters of a pan of brownies is left. Johnny eats $\frac{1}{2}$ of what is left. What fraction of a pan of brownies did Johnny eat?

31. Write a simple story problem for

$$\frac{3}{4} \cdot \frac{3}{5}.$$

Use your story problem and use pictures to explain clearly why it makes sense that the answer to the fraction multiplication problem is

$$\frac{3 \cdot 3}{4 \cdot 5}.$$

In particular, explain why the numerators are multiplied and why the denominators are multiplied.

32. For each of the following story problems, write the corresponding division problem, state which interpretation of division is involved (the *how many groups?* or the *how many in each group?*, with or without remainder), and solve the problem.

    (a) Given that 1 quart is 4 cups, how many quarts of water is 35 cups of water?

    (b) If your car used 15 gallons of gasoline to drive 330 miles, then how many miles per gallon did your car get?

    (c) If you drove 240 miles at a constant speed and if it took you $3\frac{1}{2}$ hours, then how fast were you going?

(d) Given that 1 inch is 2.54 centimeters, how tall in inches is a woman who is 153 cm tall?

(e) Will needs to cut a piece of wood .67 of an inch thick, or just a little less thick. Will's ruler shows sixteenths of an inch. How many sixteenths of an inch thick should Will cut his piece of wood?

33. Make up and solve three different story problems for $9 \div 4$.

(a) In the first story problem, the answer should best be expressed as 2, remainder 1.

(b) In the second story problem, the answer should best be expressed as $2\frac{1}{4}$.

(c) In the third story problem, the answer should best be expressed as 2.25.

34. (a) Explain why $12 \div 0$ is not defined by rewriting the problem $12 \div 0 = ?$ as a multiplication problem.

(b) Explain why $12 \div 0$ is not defined by writing a story problem for $12 \div 0$.

35. Maya is working on the division problem $245 \div 15$. Maya's work appears in Figure 10

$$\begin{array}{r} 15 \\ \times 2 \\ \hline 30 \end{array} \qquad 8 \times 30 = 240 \qquad 8 \times 2 = 16 \qquad \boxed{16 \text{ R } 5}$$

Figure 10: Maya's work for $245 \div 15$

(a) Explain why Maya's strategy makes sense. It may help you to work with a story problem for $245 \div 15$.

(b) Write equations that correspond to Maya's work and that demonstrate that $245 \div 15 = 16$, remainder 5. (One side of each equation should be 245.)

36. (a) Write a story problem for $3458 \div 6$ using the *how many groups?* interpretation of division.

(b) Use the scaffold method to calculate $3458 \div 6$. Interpret each step in the scaffold method in terms of your story problem.

Sources:

Mathematics for Elementary Teachers, volume 1, preliminary edition, by Sybilla Beckmann, Addison-Wesley, 2003

Mathematics for Elementary Teachers Instructor's Manual by Sybilla Beckmann, Addison-Wesely, 2004 expected