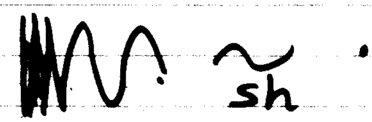
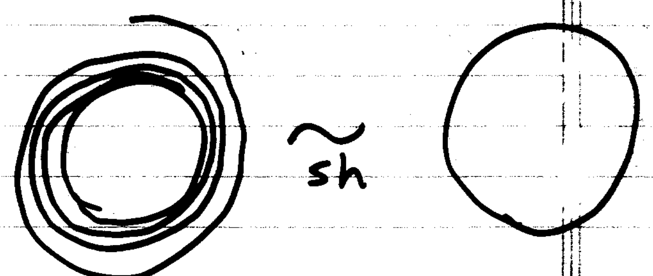
Def $X \underset{sh}{\simeq} Y$ if :

when $X, Y \underset{tame}{\hookrightarrow} [0,1]^{\infty}$,

they have htpy. equiv. nbhd sequences (mod shifts)

Example





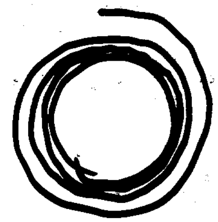
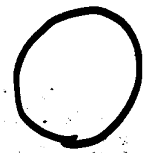
Def $f : X \to Y$ is **cell-like** if

$f^{-1}(y) \underset{sh}{\sim} pt$ for every $y \in Y$.

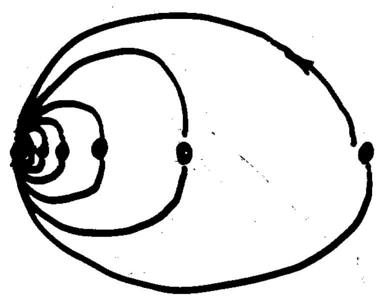
Def $X \underset{ce}{\simeq} Y$ if $\exists$ cell-like maps

$$X \swarrow Z_1 \searrow Z_2 \swarrow Z_3 \searrow \cdots \swarrow Z_{2n-1} \searrow Y$$

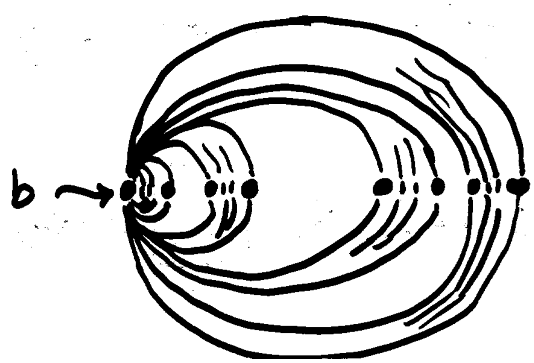
Fact: $X \underset{ce}{\sim} Y \Rightarrow X \underset{sh}{\sim} Y$

Example (Ferry):  $\underset{ce}{\not\sim}$ 

$\mathcal{H}$ = Hawaiian earing

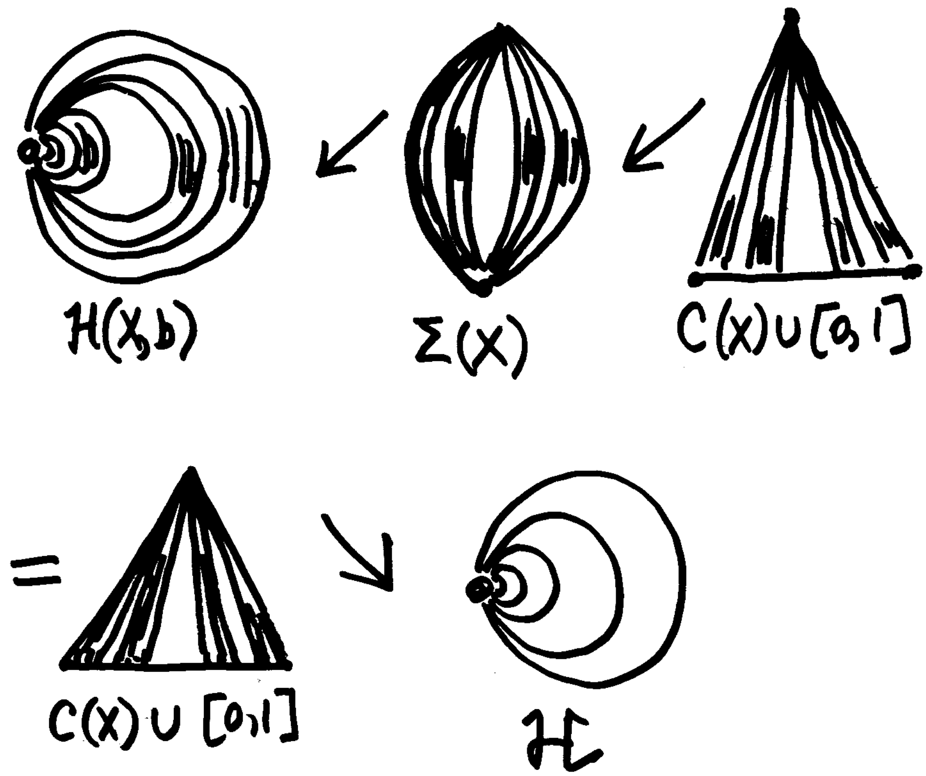


For $b \in X$ cpct. tot. disc $\subset [0,1]$,
$\mathcal{H}(X,b)$ = generalized Hawaiian earing



FACT: $\forall\, b \in X$ cpct. tot. disc $\subset [0,1]$,

$$\mathcal{H}(X, b) \underset{ce}{\simeq} \mathcal{H}$$



(Daverman - Venema)

<u>Def</u> X is a CAT(0) space if

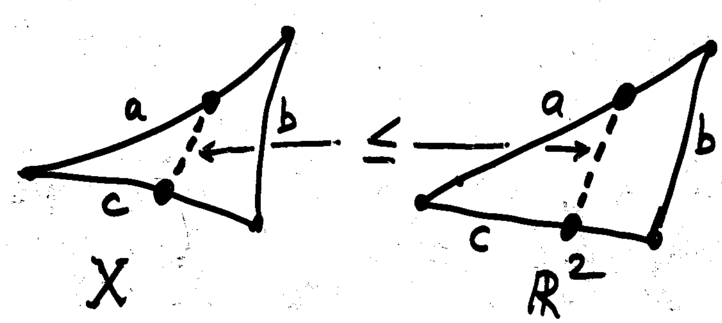
- X is a geodesic metric space

($\forall\, p, q \in X$, if $d = d(p,q)$, then $\exists$ isometric embedding $[0,d] \to X$ joining p to q)

- X is proper

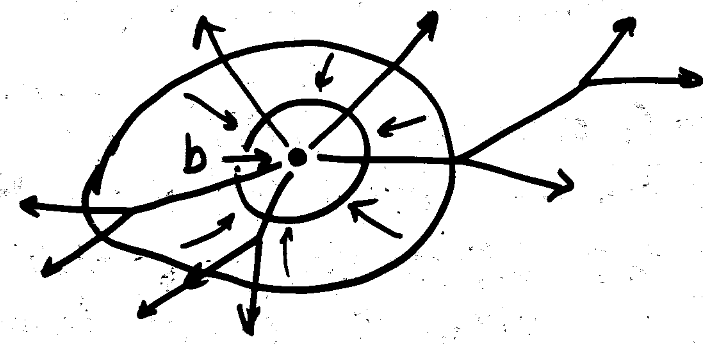
(closed metric balls are compact)

- Distances in geodesic Δ's in X are dominated by distances in comparison Δ's in $\mathbb{R}^2$



Each CAT(0) space X has a **visual boundary**

$$\partial X = \{\text{rays emanating from } b\} = \varprojlim_{r \to \infty} \partial B_r(b)$$



G is a CAT(0) group if it acts *geometrically* on a CAT(0) space

- prop. disc
- cocompactly
- by isometry

If G is a CAT(0) group and G acts geom. on a CAT(0) space X, then ∂X is called a boundary of G.

Topology of $\partial G \leftrightsquigarrow$ Algebra of G

- $\text{c-dim}_{\mathbb{Z}} G = \dim(\partial G) + 1$ (Bestvina)
- G one-ended CAT(0) and ∂G has global cut point $\Rightarrow$ $G \supset$ inf. torsion group that fixes the point (Swenson)

Theorem (Gromov) Hyperbolic groups have unique boundaries

Idea:

G acts geom. on hyp. spaces X, Y ⇒

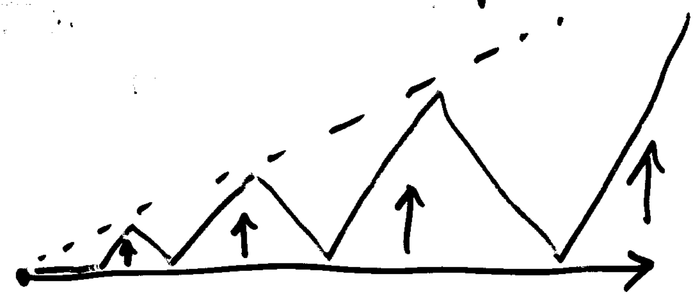
∃ quasi-isometry $f: X \to Y$ ⇒

f induces a bijection:

{rays in X from b} → {rays in Y from f(b)}

Doesn't work in CAT(0) spaces:

quasi-isometries don't preserve rays



Example The Croke-Kleiner group

$\Gamma = (a, b, c, d \mid [a,b] = [b,c] = [c,d] = 1)$

is a CAT(0) group with non-homeomorphic boundaries

Theorem (Bestvina) All boundaries of a given CAT(0) group are shape equivalent

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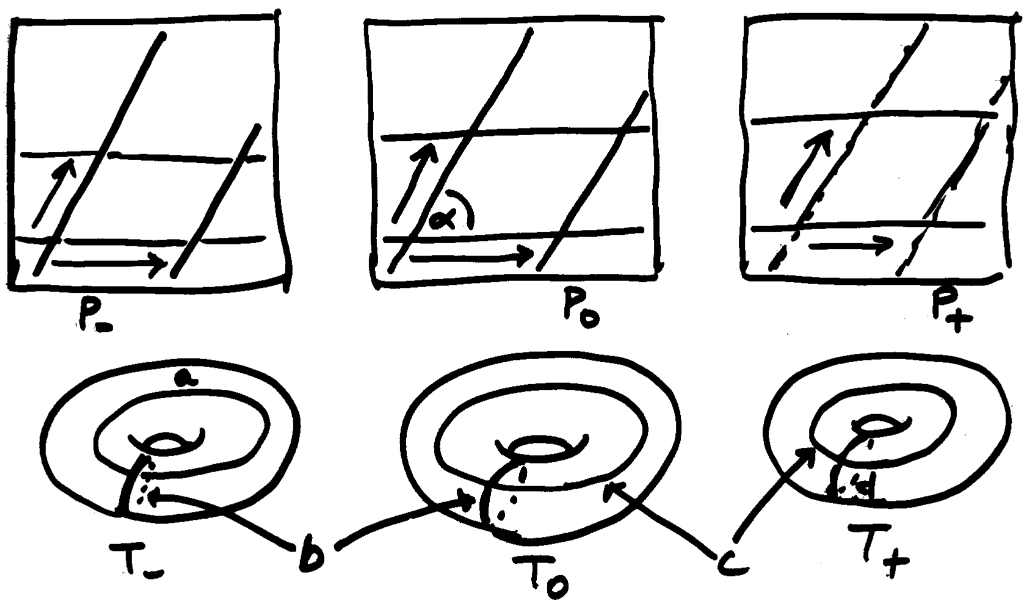
Conjecture (Bestvina) All boundaries of a given CAT(0) group are cell-like equivalent.

Theorem: All known crows are black.

Theorem (Ancel-Guilbault-Wilson) All known boundaries of the Croke-Kleiner group are cell-like equivalent (to $\mathcal{H}$).

The Croke-Kleiner Example

Let $0 < \alpha \le \pi/2$



$$X_\alpha = T_- \cup_b T_0 \cup_c T_+$$

$\pi : \tilde{X}_\alpha \to X_\alpha$ univ cover

(lift metric)

$\Gamma = \pi_1(X_\alpha)$ (indep of α)

Γ acts geom on $\tilde{X}_\alpha$

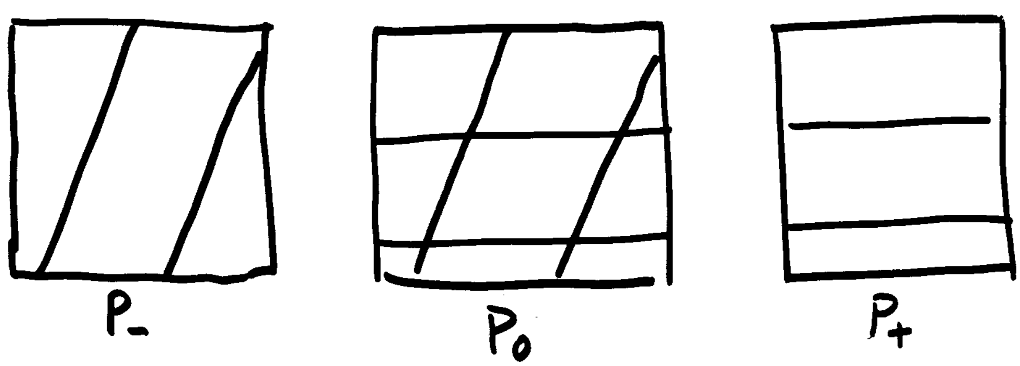
Theorem (Croke–Kleiner)

If $\alpha < \pi$, then $\partial\tilde{X}_\alpha \not\cong \partial\tilde{X}_{\pi/2}$.

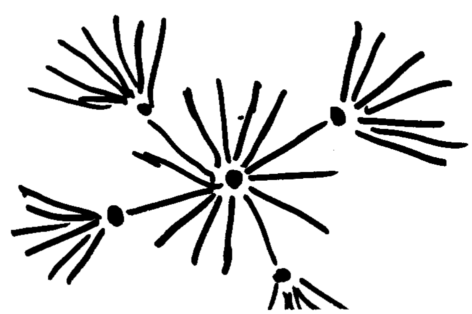
Theorem (J. Wilson)

If $0 < \alpha < \beta \leq \pi$, then $\partial\tilde{X}_\alpha \not\cong \partial\tilde{X}_\beta$.

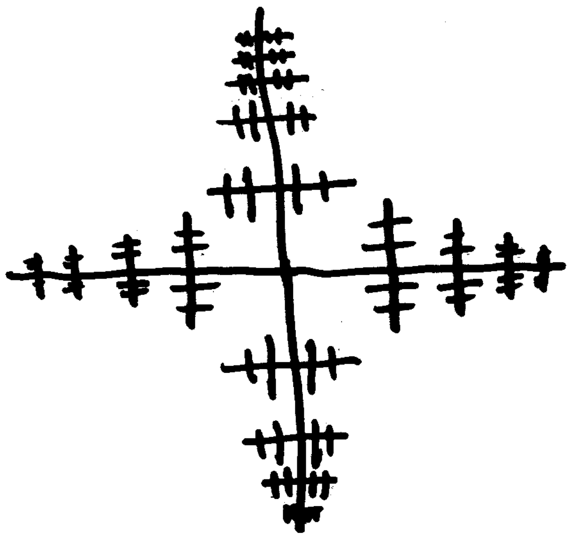
$\tilde{X}_\alpha$ = ctble union of planes of 3 types:



Schematic:



A component of $\pi^{-1}(T_- \cup T_0)$ (or $\pi^{-1}(T_0 \cup T_+)$) is $\approx$

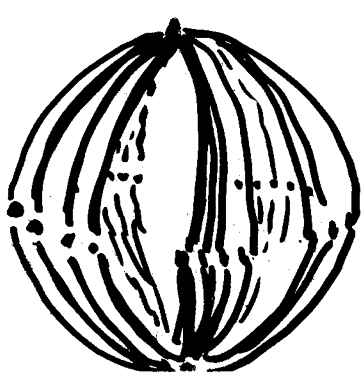


$\times \mathbb{R}$

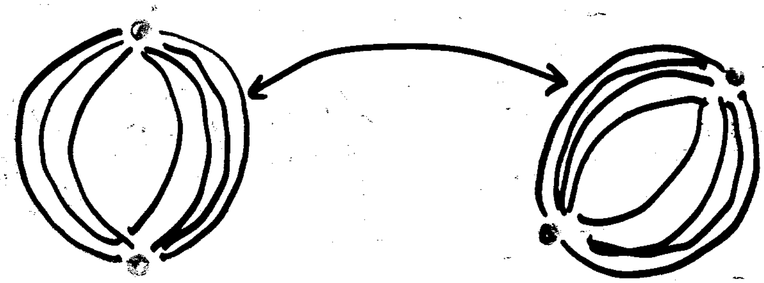
(univ cover of figure 8)

∂(component of $\pi^{-1}(T_+ \cup T_0)$)

$\approx \Sigma$(Cantor Set)



Two adjacent Σ(Cantor Set)'s are glued together along circles with an α twist



The union of Σ(Cantor Set)'s is a dense subset of $\partial \tilde{X}_\alpha$.

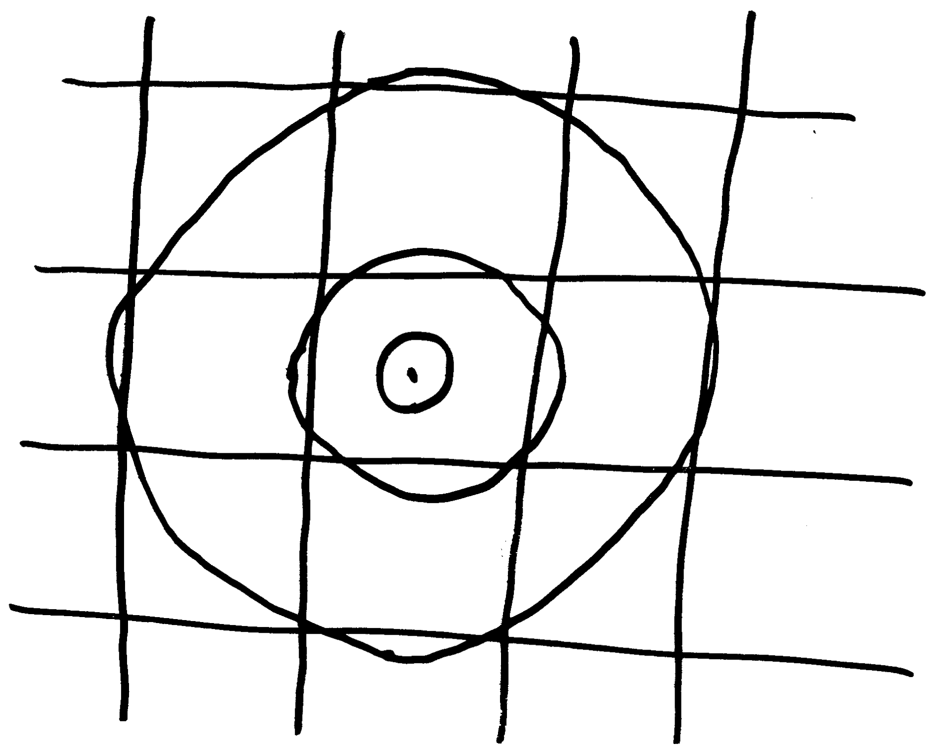
Let $0 < r_1 < r_2 < r_3 \cdots \to \infty$

Let $g_n : \partial B_{r_{n+1}}(b) \longrightarrow \partial B_{r_n}(b)$

be geodesic retraction

Then $\partial \tilde{X}_\alpha = \varprojlim \{\partial B_{r_n}, g_n\}$

Look at $\partial B_{r_n}(b)$ in P_0 :



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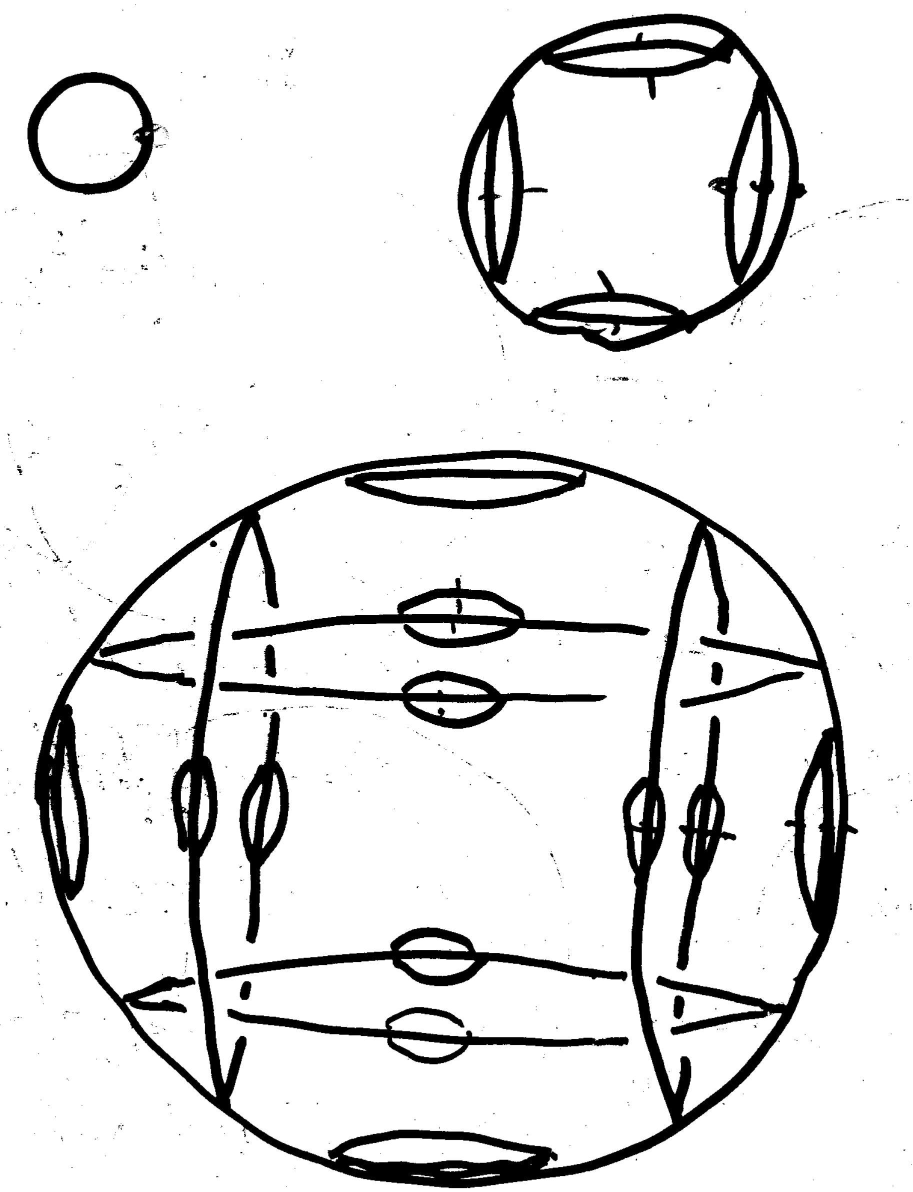
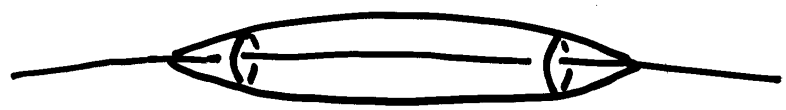
Actual $\partial B_{r_n}(b)$:



Eventually see:

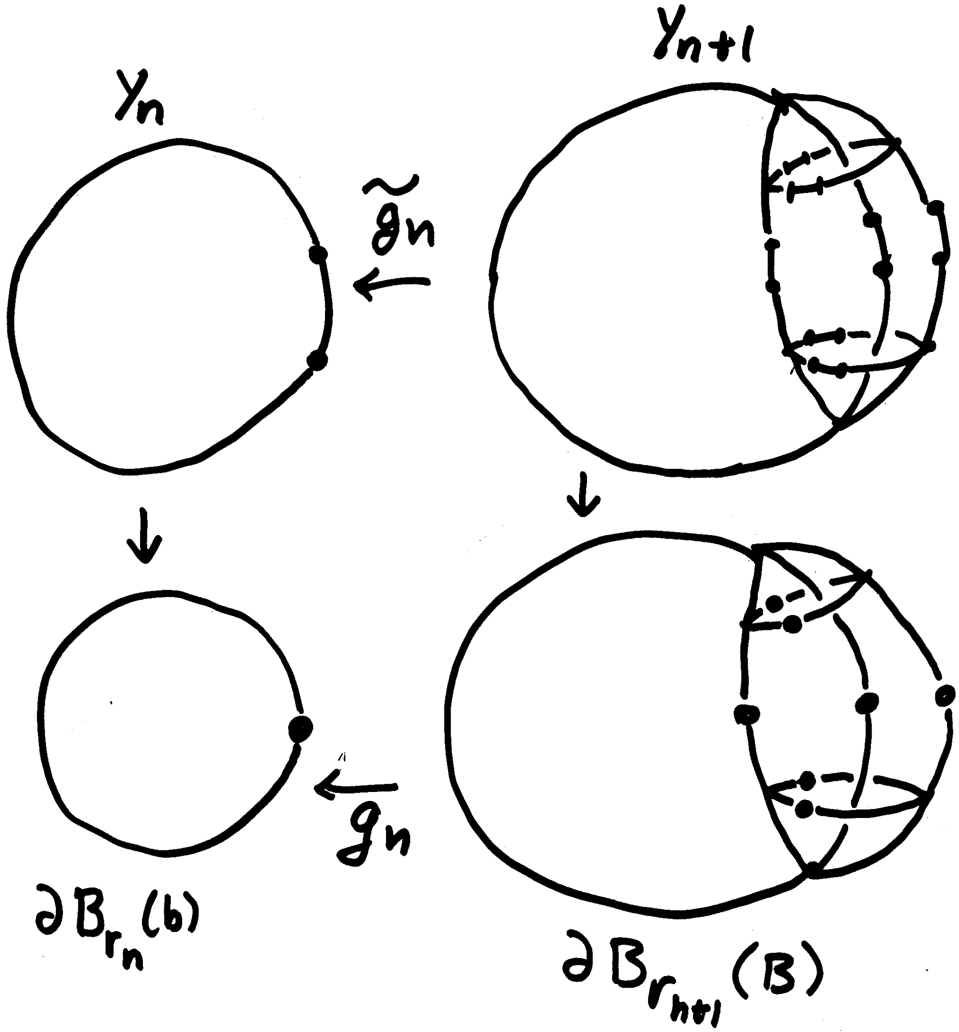


$\{\partial B_{r_n}(b), g_n\}$ has the properties:

- $\partial B_{r_n}(b)$ is a finite 1-complex
- $\exists$ finite set $A_n \subset \partial B_{r_n}(b)$ containing no vertices such that
- $\partial B_{r_n}(b) - A_n$ is contractible
- $g_n^{-1}(A_n) \subset A_{n+1}$

Lemma. The inverse limit of any such inverse system is cell-like equivalent to $\mathcal{H}$.

Proof: Blow up points of A_n to arcs



$$\begin{array}{ccccccc} Y_1 & \leftarrow & Y_2 & \leftarrow & \cdots & & Y_\infty \\ \downarrow & & \downarrow & & & & \downarrow \sim \\ \partial B_{r_1}(b) & \leftarrow & \partial B_{r_2}(b) & \leftarrow & \cdots & & \partial X_\alpha \end{array}$$

$$Y_i = \underbrace{C_i}_{\text{contractible}} \cup \underbrace{\text{arcs}}_{\text{finitely many}}$$

$$Y_\infty = C \cup \underbrace{\text{arcs}}_{\text{cantor set's worth}}$$

$C \underset{sh}{\sim} pt$

$$\begin{array}{c} Y_i \\ \downarrow \text{ cell-like} \\ \partial B_{r_i}(b) \end{array} \rightsquigarrow \begin{array}{c} Y_\infty \\ \downarrow\sim \text{ cell-like} \\ \partial X_\alpha \end{array}$$

$$Y_\infty \xrightarrow[\text{like}]{\text{cell}} Y_\infty / C \approx \mathcal{H}(X) \underset{\text{c.e.}}{\sim} \mathcal{H}. \quad \square$$

QUESTIONS:

- Is every C-K group boundary already known?
- Is every C-K group boundary cell-like equivalent to $\mathcal{H}$?
- Are any two C-K group boundaries equivariantly cell-like equivalent through C-K group boundaries?
- If $\partial_1\Gamma$ and $\partial_2\Gamma$ are C-K group boundaries, is there a C-K group boundary $\partial_3\Gamma$ and equivariant cell-like maps

$$\partial_1\Gamma \swarrow \partial_3\Gamma \searrow \partial_2\Gamma\ ?$$

- Is there a maximal C-K group boundary w.r.t. equivariant cell-like maps? ($\partial_{max}\Gamma \to \partial\Gamma$)