On the Finiteness of Tight Contact Structures

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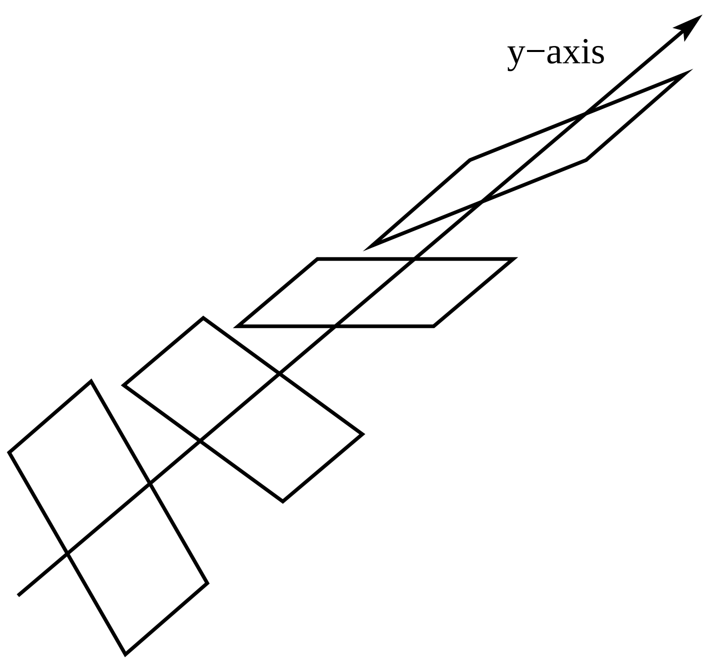
1. Introduction

M^3 - oriented, compact 3-manifold.
$\xi \subset TM$ - oriented, positive contact structure.
$\xi = \ker(\alpha)$, α 1-form, $\alpha \wedge d\alpha > 0$.

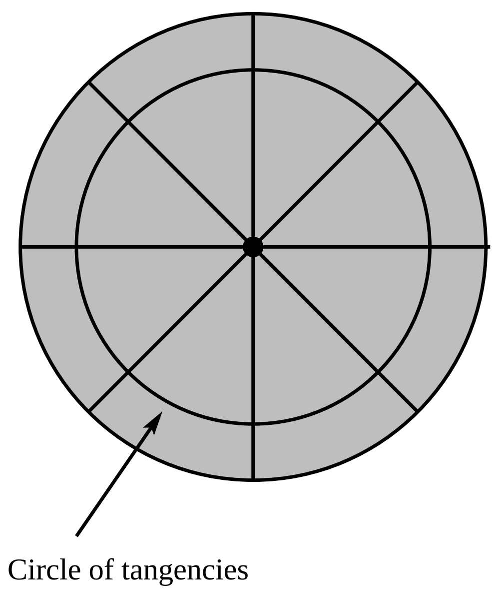
Locally, every (M, ξ) looks like the following:

Example: $(\mathbf{R}^3, \xi)$, $\alpha = dz - ydx$.
Think in terms of the "propeller picture":



Principal Tool: Embedded surfaces $\Sigma \subset M$. $\Sigma \cap \xi$ is called the **characteristic foliation** of Σ. It is a singular foliation on Σ.

An embedded disk $D \subset M$ is **overtwisted** if ξ is everywhere tangent to D along ∂D. A typical characteristic foliation $D \cap \xi$ on D is a singular foliation as below, where the center is an elliptic singular point, and ∂D^2 is a Legendrian (horizontal) curve of ξ.



Eliashberg: There is a dichotomy between overtwisted structures and tight structures.

Overtwisted = $\exists$ overtwisted disk

Tight = no overtwisted disk

1. The classification of overtwisted contact structures on M is identical to the homotopy classification of 2-plane fields on M, provided M is closed.
2. Tight structures are rather mysterious.

Let $Tight(M)$ be the space of tight contact 2-plane fields on M. Can we at least say whether $|\pi_0(Tight(M))|$ is finite or infinite?

Remark: Gray's Theorem states that if there is a homotopy ξ_t, $t \in [0,1]$, of contact structures, then there is an isotopy $\phi_t : (M, \xi_0) \stackrel{\sim}{\to} (M, \xi_t)$.

2. Main Results

Previous results for M closed and oriented:

Theorem 1 (Eliashberg) *Only finitely many cohomology classes in* $H^2(M, \mathbf{Z})$ *are Euler classes for tight contact structures.*

This is the contact analogue of the corresponding theorem for taut foliations proved by Thurston. The gist of both theorems is the following inequality:

$$|\langle e(\xi), \Sigma \rangle| \leq 2g - 2.$$

Here Σ is a closed surface of genus $g \geq 1$.

Using Seiberg-Witten theory:

Theorem 2 (Kronheimer-Mrowka) *There exist finitely many homotopy classes of 2-plane fields which carry weakly symplectically semi-fillable contact structures.*

"Symplectically fillable" = "bounds" a symplectic 4-manifold (X, ω). A corollary of the above theorem is the following, proved using results of Eliashberg and Thurston on perturbing taut foliations into semi-fillable contact structures:

Corollary 1 *There exist finitely many homotopy classes of 2-plane fields which carry taut foliations.*

Later, Gabai gave a purely 3-dimensional proof of this result.

New results:

Theorem 3 *Let M be a closed, oriented 3-manifold. Then there are finitely many homotopy classes of 2-plane fields which carry tight contact structures.*

Remark: This is a genuine improvement of the Kronheimer-Mrowka theorem because there are examples of tight contact structures which are not fillable (Etnyre-H).

There is a reason why we cannot prove finiteness of $|\pi_0(Tight(M))|$.

Example: $(T^3 = \mathbf{R}^3/\mathbf{Z}^3, \xi_n)$, given by

$$\alpha_n = \sin(2\pi nz)dx + \cos(2\pi nz)dy, n \in \mathbf{Z}^+.$$

Kanda & Giroux (independently) showed that

1. the ξ_n are distinct, and
2. every tight contact structure is diffeomorphic to one of the ξ_n.

Conjecture: Every closed, oriented, irreducible 3-manifold M which is toroidal carries infinitely many isotopic tight contact structures. Every atoroidal M carries finitely many tight contact structures.

Definition: *Toroidal* = $\exists$ torus $T \subset M$ for which $\pi_1(T) \hookrightarrow \pi_1(M)$. Such a torus T is called *incompressible.* *Atoroidal* = not toroidal.

Theorem 4 (Colin, H.-Kazez-Matić) *Let M be a closed, oriented, irreducible, toroidal 3-manifold. Then M carries infinitely many nonisomorphic contact structures.*

On the other hand:

Theorem 5 *If M is a closed, oriented, atoroidal 3-manifold, then $|\pi_0(Tight(M))|$ is finite.*

There are also relative versions of this theorem.

Boundary conditions: We assume ∂M is *convex* with characteristic foliation $\mathcal{F}$.

Theorem 6 *If M is a compact, oriented, atoroidal 3-manifold, and ∂M has no torus components, then $|\pi_0(Tight(M, \mathcal{F}))|$ is finite.*

Another result which can be proved with more work:

Theorem 7 *Consider (S^3, ξ_{std}). Fix an oriented topological knot type K. Then the set $\mathcal{L}(K, tb, r)$ of isotopy classes of Legendrian knots of type K with fixed Thurston-Bennequin invariant tb and rotation number r is finite.*

3. Sketch of Proof

The proof uses the classical approach of normalizing tight contact structures with respect to a fixed triangulation τ of M. (Haken-Kneser normal form.)

Step 1. For each $\xi \in Tight(M)$, assume τ has been isotoped so that the 1-skeleton is a Legendrian graph and each face of τ is a convex surface.

A **convex surface** Σ is an embedded surface, closed or compact with Legendrian boundary, for which $\exists$ a contact vector field $v \pitchfork \Sigma$.

Step 1 is easily accomplished because:

1. Every embedded graph can be C^0-approximated by a Legendrian graph.

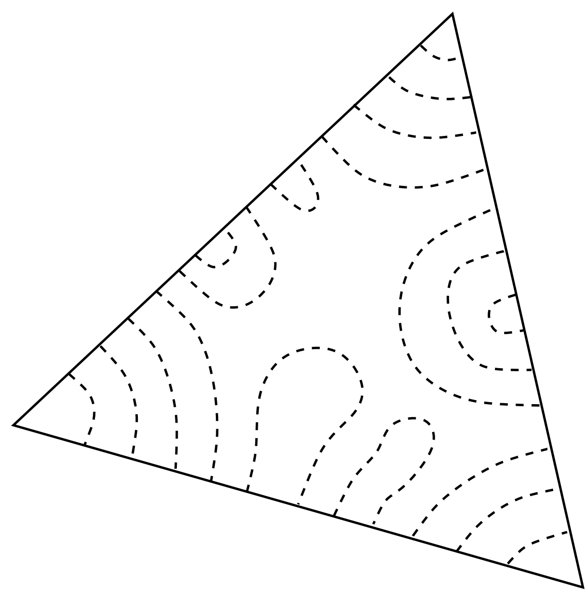
2. (Giroux) A C^∞-generic surface is convex.

In contact topology, one only needs to remember the **dividing multicurve** Γ_Σ of a convex surface Σ. Γ_Σ is a properly embedded, smooth multicurve consisting of the set of points where ξ is normal to Σ, measured with respect to v.

Proposition 1 (Giroux's criterion) *A convex surface Σ has a tight neighborhood iff*

1. *$\Sigma \neq S^2$ and Γ_Σ has no homotopically trivial closed curves, or*
2. *$\Sigma = S^2$ and $\#\Gamma_\Sigma = 1$.*

Example:

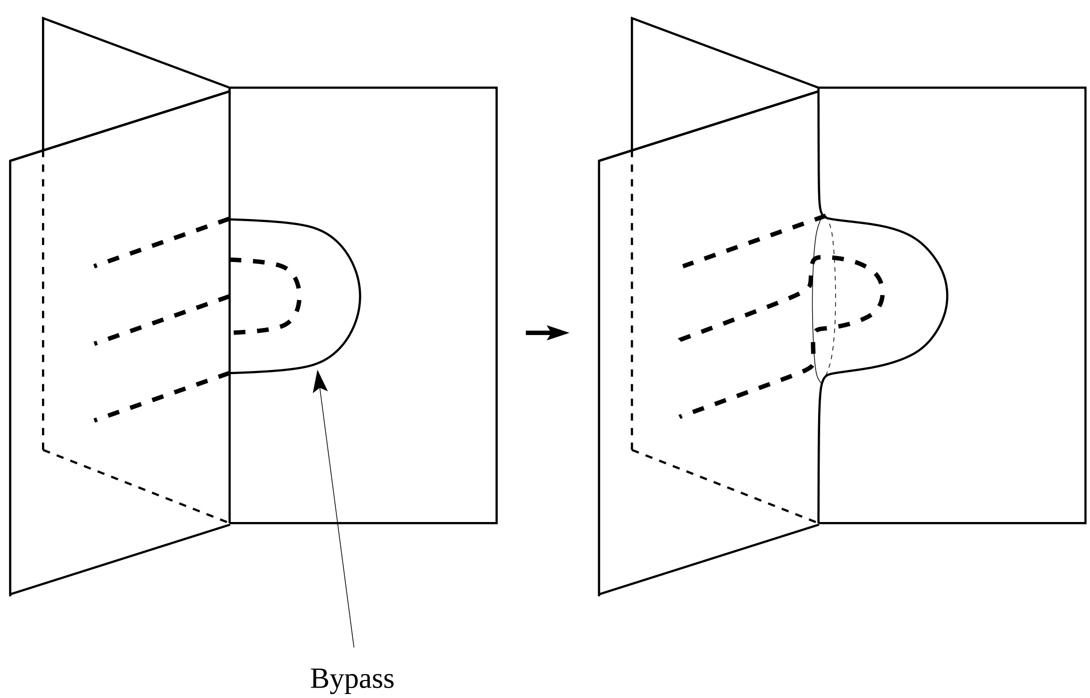


Remark: It suffices to control the restriction of ξ to the 2-skeleton.

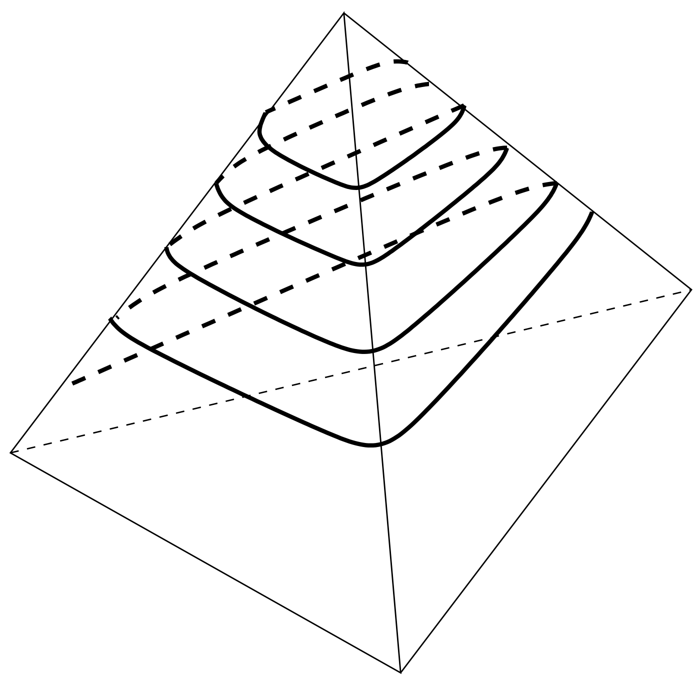
Theorem 8 (Eliashberg) *Let Γ_{S^2} be a dividing set with $\#\Gamma_{S^2} = 1$, and let $\mathcal{F}$ be a characteristic foliation adapted to Γ_{S^2}. Then there exists a unique tight contact structure up to isotopy on B^3 with boundary condition $\mathcal{F}$ on $\partial B^3 = S^2$.*

Step 2. Simplify the characteristic foliation on each face.

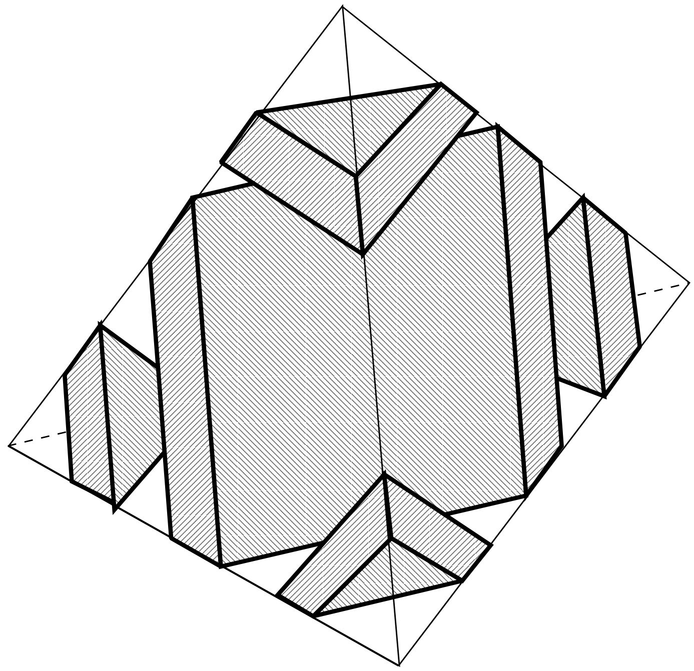
First simplification: Get rid of all the edge-parallel dividing arcs.



After some modifications: Can assume that the holonomy around each vertex is -2.



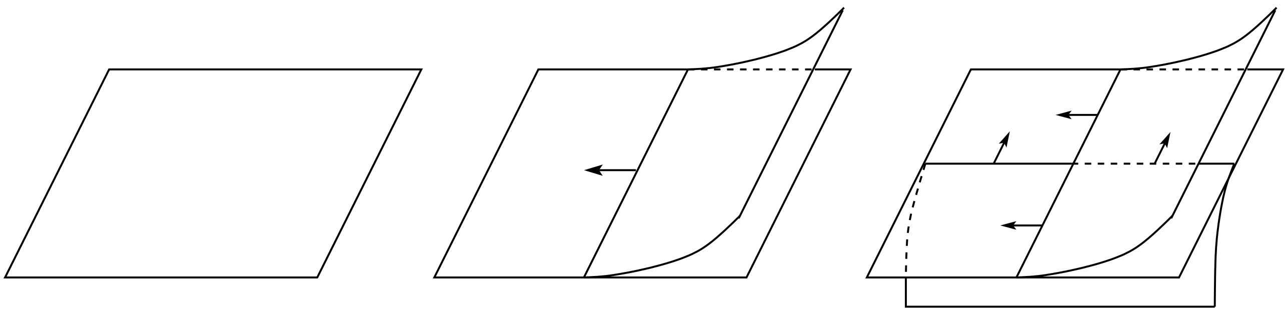
We can isolate all the nonfiniteness inside at most 5 prisms inside each 3-simplex.



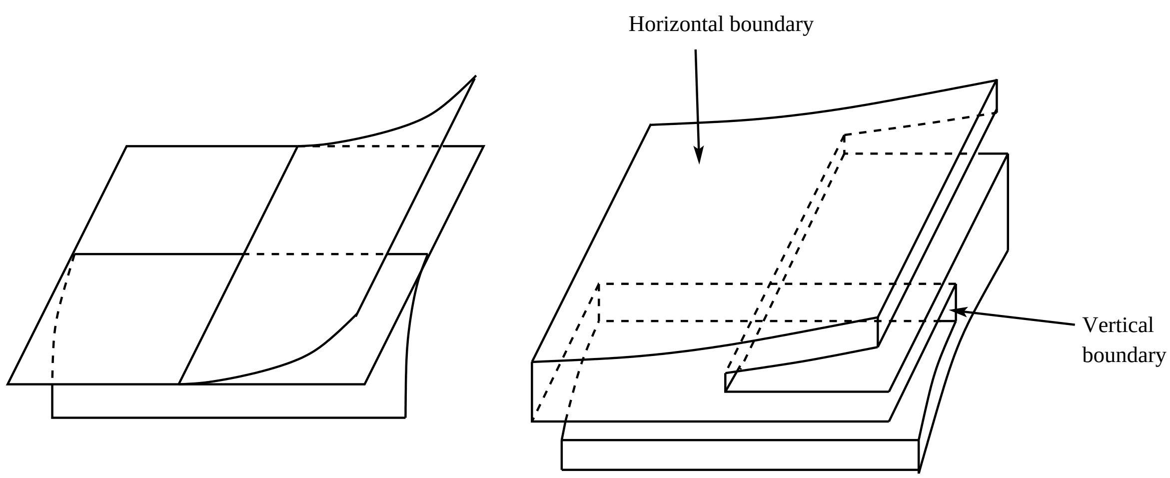
Each prism is $\Delta \times [0,1]$ where Δ is a triangle or rectangle. We may assume the prism admits a Legendrian fibration $\{pt\} \times [0,1]$.

Step 3. Make a fibered nbhd. of branched surface $\mathcal{B}$ by gluing together the prisms. All the "infiniteness" is contained inside the branched surface.

Definition: A **branched surface** is locally made out of the following pieces:



A **fibered nbhd** $N(\mathcal{B})$:



Thus, every tight contact structure ξ (up to isotopy) is "generated by" $(\mathcal{B}_1, \zeta_1), \cdots, (\mathcal{B}_k, \zeta_k)$, i.e.,

1. $\xi|_{M \setminus N(\mathcal{B}_i)} = \zeta|_{M \setminus N(\mathcal{B}_i)}$ for some i.

2. ξ is tangent to the fibers of $N(\mathcal{B}_i)$.

The finiteness of homotopy follows quickly from a generalization of the following example:

Example: (T^3, ξ_n) revisited. ξ_n given by

$$\alpha_n = \sin(2\pi nz)dx + \cos(2\pi nz)dy$$

but we can also isotop by taking

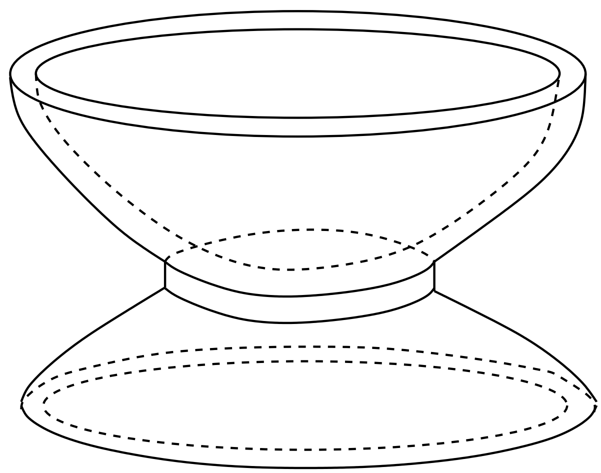
$$\alpha_n(t) = tdz + \alpha_n.$$

In the limit as $t \to \infty$, $\xi_n(t) \to ker(dz)$.

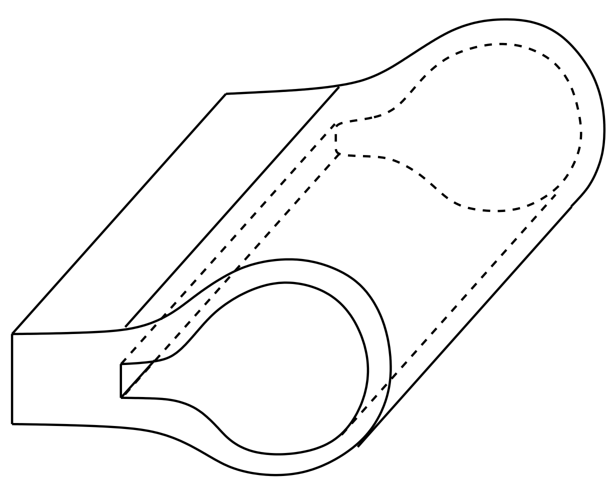
Step 4. Now assume M is **atoroidal**. We verify that every $\mathcal{B}_i$ constructed above, after possible simplification, satisfies the conditions of an **incompressible branched surface**:

1. Every component of the horizontal boundary of $N(\mathcal{B}_i)$ is incompressible in $M \setminus N(\mathcal{B}_i)$.

2. $\nexists$ disks of contact.



3. $\nexists$ monogons.



Theorem 9 (Floyd-Oertel) *Every closed surface fully carried by an incompressible branched surface is incompressible.*

1. After possibly shrinking $\mathcal{B}_i$, if $\mathcal{B}_i$ is still nonempty, it will fully carry a closed surface.

2. Every closed surface carried by $\mathcal{B}_i$ is a torus or a Klein bottle. (Klein bottle case treated separately.)

3. However, M cannot have an incompressible torus by assumption.

This implies that $\mathcal{B}_i$ is empty. QED.