

Projective maps and symplectic invariants

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Linear systems on symplectic manifolds

(X^{2n}, ω) symplectic, compact

- $\frac{1}{2\pi}[\omega] \in H^2(X, \mathbb{Z})$ (not restrictive)
- J compatible with ω ; $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$
- L line bundle such that $c_1(L) = \frac{1}{2\pi}[\omega]$
- ∇^L , curvature $-i\omega$

$L =$ “almost-complex ample line bundle”.

\Rightarrow approximately holomorphic sections of $L^{\otimes k}$, $k \gg 0$.

Sections with transversality properties (“generic linear systems”) \Rightarrow topological structures on symplectic manifolds.

Theorem 1. (Donaldson) *For $k \gg 0$, two suitably chosen A.H. sections of $L^{\otimes k}$ endow X with a structure of symplectic Lefschetz pencil, canonical up to isotopy.*

Theorem 2. *For $k \gg 0$, three suitable A.H. sections of $L^{\otimes k}$ define a singular fibration $X \rightarrow \mathbb{C}\mathbb{P}^2$ with generic local models, canonical up to isotopy.*

Generally: A.H. sections $s_{k,0}, \dots, s_{k,m} \in \Gamma(L^{\otimes k})$

\Rightarrow approx. holomorphic maps $f_k : X \rightarrow \mathbb{C}\mathbb{P}^m$

Need estimated transversality for the jets of these maps.

Estimated transversality of jets

$E_k = \mathbf{C}^{m+1} \otimes L^{\otimes k}$ asympt. very ample vector bundles,
 holom. jet bundles $\mathcal{J}^r E_k = \bigoplus_{j=0}^r (T^* X^{(1,0)})_{\text{sym}}^{\otimes j} \otimes E_k$.

$$s_k \in \Gamma(E_k) \Rightarrow j^r s_k = (s_k, \partial s_k, (\partial \partial s_k)_{\text{sym}}, \dots).$$

$\mathcal{S}_k =$ finite asympt. holomorphic stratifications of $\mathcal{J}^r E_k$:
 (Whitney, transverse to the fibers, geometry estimates).

The strata $\mathcal{S}_k^{(i)}$ enumerate the possible singular behaviors:
 e.g. for Lefschetz pencils, $\{s_k = 0\}$ and $\{s_k \neq 0, \partial f_k = 0\}$.

The jet $j^r s_k$ is η -transverse to \mathcal{S}_k if
 $\text{dist}(j^r s_k(x), \mathcal{S}_k^{(i)}) < \eta \Rightarrow$ the graph of $j^r s_k$ is transverse at x
 to $T\mathcal{S}_k^{(i)}$, with minimum angle $> \eta$.
 (if $\text{codim } \mathcal{S}_k^{(i)} > n$ then $j^r s_k$ remains away from $\mathcal{S}_k^{(i)}$).

Theorem 3. \mathcal{S}_k A.H. stratifications of $\mathcal{J}^r E_k$; $\delta > 0$;

σ_k A.H. sections of E_k

\Rightarrow for large enough k , \exists A.H. sections s_k of E_k s.t.

- (1) $|s_k - \sigma_k|_{C^{r+1}, g_k} < \delta$;
- (2) $j^r s_k$ is $\eta(\delta)$ -transverse to \mathcal{S}_k .

After slightly perturbing $\bar{\partial}\sigma_k$, one gets maps $X \rightarrow \mathbf{C}\mathbf{P}^m$ lo-
 cally modelled on generic holomorphic maps.

Moreover, 1-parameter version for $(J_t)_{t \in [0,1]}$, $(s_{k,t})_{k \gg 0, t \in [0,1]}$

\Rightarrow the construction is canonical for large k .

Symplectic Lefschetz pencils

(X^{2n}, ω) symplectic, $s_0, s_1 \in \Gamma(L^{\otimes k})$ suitably chosen
 \Rightarrow symplectic Lefschetz pencil (Donaldson) :

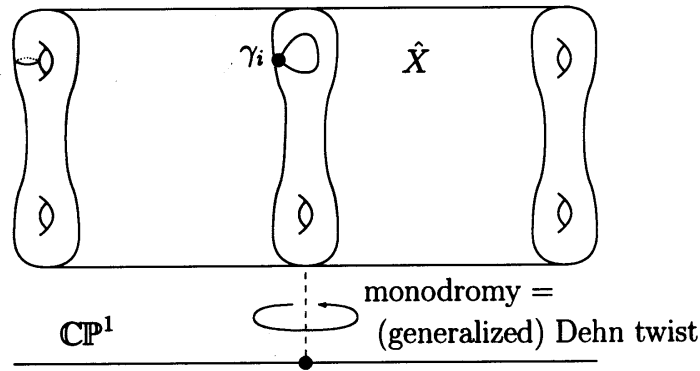
$$\Sigma_\alpha = \{x \in X, s_0 + \alpha s_1 = 0\} \quad (\alpha \in \mathbb{C}\mathbb{P}^1)$$

symplectic hypersurfaces, smooth except for finitely many singular points, intersecting at the base locus $Z = \{s_0 = s_1 = 0\}$ (codim. 4).

Projective map $f = (s_0 : s_1) : X - Z \rightarrow \mathbb{C}\mathbb{P}^1$:

local model $f(z) = z_1^2 + \dots + z_n^2$ near critical points.

Blow up $Z \Rightarrow$ Lefschetz fibration $\hat{X} \rightarrow \mathbb{C}\mathbb{P}^1$



$$\text{Monodromy} = \theta : \pi_1(\mathbb{C} - \{\text{pts}\}) \rightarrow \text{Map}^\omega(\Sigma^{2n-2}, Z)$$

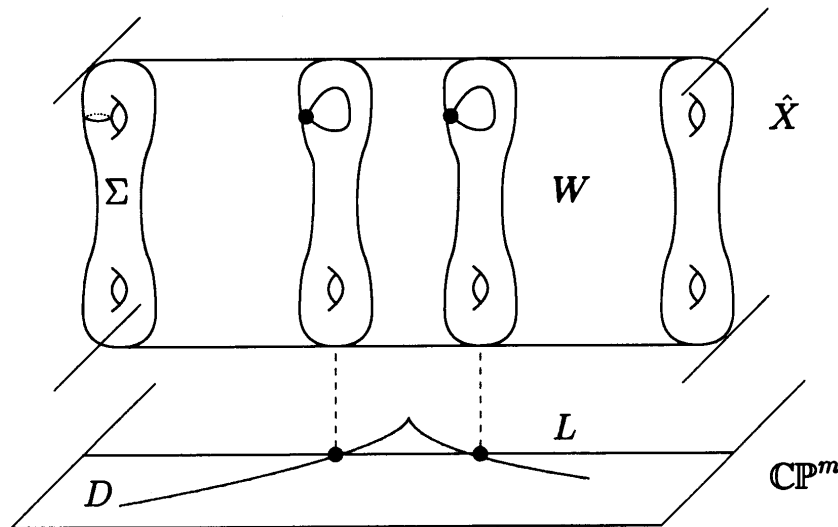
$$\text{Map}^\omega(\Sigma, Z) := \pi_0(\{\phi \in \text{Symp}(\Sigma, \omega), \phi|_{U(Z)} = \text{Id}\})$$

X can be recovered up to symplectomorphism from the fiber and the monodromy map (Gompf).

The higher dimensional case

(X^{2n}, ω) symplectic, $s_0, \dots, s_m \in \Gamma(L^{\otimes k})$ well-chosen
 \Rightarrow projective map $f = (s_0 : \dots : s_m) : X - Z \rightarrow \mathbb{C}P^m$.

- Fibers = codimension $2m$ symplectic submanifolds, intersecting at the base locus Z (codim. $2m + 2$).
- Blow up along $Z \Rightarrow$ singular fibration $\hat{X} \rightarrow \mathbb{C}P^m$.
- Critical set: $R^{2m-2} \subset X$ stratified symplectic submfd. (= all singular points of all fibers)
- $D = f(R) \subset \mathbb{C}P^m$ singular symplectic hypersurface.
- Monodromy $\theta : \pi_1(\mathbb{C}P^m - D) \rightarrow \text{Map}^\omega(\Sigma, Z)$.
- For a generic line $L \subset \mathbb{C}P^m$, $W = f^{-1}(L)$, $f|_W : \hat{W} \rightarrow \mathbb{C}P^1$ is a SLP, monodromy θ .



The higher dimensional case

The fiber $(\Sigma, \omega|_{\Sigma})$, the topology of the singular hypersurface $D \subset \mathbb{C}\mathbb{P}^m$, and the monodromy map $\theta : \pi_1(\mathbb{C}^m - D) \rightarrow \text{Map}^{\omega}(\Sigma, Z)$ determine (X, ω) up to symplectomorphism.

- For $m > 2$, the topology of $D \subset \mathbb{C}\mathbb{P}^m$ is not understood.
- For $\dim \Sigma > 2$, $\text{Map}^{\omega}(\Sigma, Z)$ is not understood.

Idea: Dimensional induction

- $f_n : X^{2n} \rightarrow \mathbb{C}\mathbb{P}^2$, discriminant curve $D_n \subset \mathbb{C}\mathbb{P}^2$
+monodromy $\theta_n : \pi_1(\mathbb{C}^2 - D_n) \rightarrow \text{Map}^{\omega}(\Sigma^{2n-4}, \dots)$.
 D_n, Σ^{2n-4} and θ_n characterize X .
- Let $W^{2n-2} = f_n^{-1}(L)$ for a generic line:
then W carries a SLP with fiber Σ and monodromy θ_n .
Adding a section, refine this SLP into a map
 $f_{n-1} : W^{2n-2} \rightarrow \mathbb{C}\mathbb{P}^2$, discriminant curve $D_{n-1} \subset \mathbb{C}\mathbb{P}^2$
+monodromy $\theta_{n-1} : \pi_1(\mathbb{C}^2 - D_{n-1}) \rightarrow \text{Map}^{\omega}(Z^{2n-6}, \dots)$
 D_n, D_{n-1}, Z^{2n-6} and θ_{n-1} characterize X .
- Iterate until get a map $f_2 : Y^4 \rightarrow \mathbb{C}\mathbb{P}^2$:
Fiber = N points, monodromy $\theta_2 : \pi_1(\mathbb{C}\mathbb{P}^2 - D_2) \rightarrow S_N$.
 D_n, D_{n-1}, \dots, D_2 and θ_2 characterize X .

So in principle it is enough to understand plane curves.

Branched covers of $\mathbb{C}\mathbb{P}^2$

(X^4, ω) symplectic, $s_0, s_1, s_2 \in \Gamma(L^{\otimes k})$ suitably chosen
 $\Rightarrow f = (s_0 : s_1 : s_2) : X \rightarrow \mathbb{C}\mathbb{P}^2$.

Local singular models near branch curve $R \subset X$:

- branched covering : $(x, y) \mapsto (x^2, y)$.

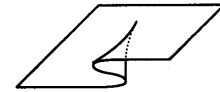
$$R : x = 0 \quad f(R) : X = 0$$



$$X^{2n} \rightarrow \mathbb{C}\mathbb{P}^2 : (z_1, \dots, z_n) \mapsto (z_1^2 + \dots + z_{n-1}^2, z_n)$$

- cusp : $(x, y) \mapsto (x^3 - xy, y)$.

$$R : y = 3x^2 \quad f(R) : 27X^2 = 4Y^3$$



$$X^{2n} \rightarrow \mathbb{C}\mathbb{P}^2 : (z_1, \dots, z_n) \mapsto (z_1^3 - z_1 z_n + z_2^2 + \dots + z_{n-1}^2, z_n)$$

R smooth symplectic curve in X .

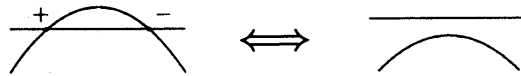
$D = f(R)$ symplectic, immersed except at the cusps.

Generic singularities :

complex cusps; nodes (both orientations)



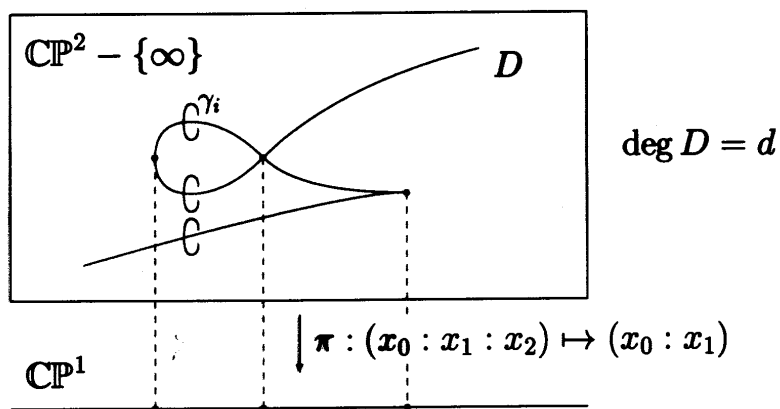
Theorem 2 \Rightarrow up to cancellation of nodes, the topology of D is a symplectic invariant (if k large).



The topology of plane curves

(Moishezon-Teicher, Auroux-Katzarkov-Yotov)

Perturbation $\Rightarrow D =$ singular branched cover of \mathbb{CP}^1 .



Monodromy $= \rho : \pi_1(\mathbb{C} - \{\text{pts}\}) \rightarrow B_d$ (braid group)

$\Rightarrow D$ is described by a “braid group factorization”
(involving cusps, nodes, tangencies).

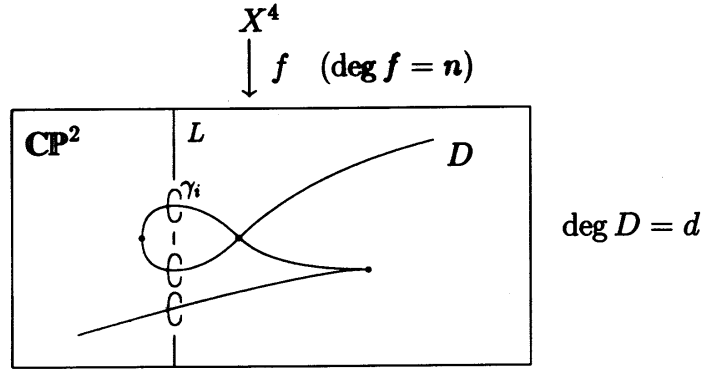
The braid factorization characterizes D completely, but cannot be used as invariant in practice.

Moishezon-Teicher: use $\pi_1(\mathbb{CP}^2 - D)$ as invariant.

$\pi_1(\mathbb{CP}^2 - D)$ is generated by “geometric generators” $(\gamma_i)_{1 \leq i \leq d}$; relations given by the braid factorization.

Problem: in the symplectic case, node cancellations affect $\pi_1(\mathbb{CP}^2 - D)$.

Stabilized fundamental groups



$L \simeq \mathbf{C} \subset \mathbf{CP}^2$ generic line, $i : L - \{p_1, \dots, p_d\} \hookrightarrow \mathbf{CP}^2 - D$
 $\Rightarrow i_* : F_d = \langle \gamma_1, \dots, \gamma_d \rangle \rightarrow \pi_1(\mathbf{CP}^2 - D)$ surjective.

Geometric generators: $\Gamma = \{\text{conjugates of } i_*\gamma_1, \dots, i_*\gamma_d\}$.

$\theta : \pi_1(\mathbf{CP}^2 - D) \rightarrow S_n$ maps elements of Γ to transpositions.

$\delta : \pi_1(\mathbf{CP}^2 - D) \rightarrow \mathbb{Z}_d$ linking number ($\delta(\gamma_i) = 1$).

Relations: for each special point, two elements of Γ s.t.

- tangency: $\gamma = \gamma'$; $\theta(\gamma)$ and $\theta(\gamma')$ identical.
- node: $\gamma\gamma' = \gamma'\gamma$; $\theta(\gamma)$ and $\theta(\gamma')$ disjoint.
- cusp: $\gamma\gamma'\gamma = \gamma'\gamma\gamma'$; $\theta(\gamma)$ and $\theta(\gamma')$ adjacent.

$K =$ normal subgroup $\langle [\gamma, \gamma'], \gamma, \gamma' \in \Gamma, \theta(\gamma), \theta(\gamma') \text{ disjoint} \rangle$.

Add a pair of nodes \Leftrightarrow quotient by an element of K .

Theorem 4. For $k \gg 0$, $G_k(X, \omega) = \pi_1(\mathbf{CP}^2 - D_k)/K_k$
and $G_k^0(X, \omega) = \text{Ker}(\theta_k, \delta_k)/K_k$ are symplectic invariants.

Horikawa surfaces

X_1 = double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched along (6, 12).

X_2 = double cover of \mathbb{F}_6 branched along $5\Delta_0 \cup \Delta_\infty$.

X_1, X_2 minimal alg. surfaces of general type.

X_1, X_2 homeomorphic, not \mathbb{C} deformation equivalent.

X_1, X_2 diffeomorphic ??? (same SW invariants)

X_1, X_2 symplectomorphic ??? (Donaldson)

ω_1 Kähler form on X_1 , $\frac{1}{2\pi}[\omega_1] = K_{X_1} = \pi^*(1, 4)$.

ω_2 Kähler form on X_2 , $\frac{1}{2\pi}[\omega_2] = K_{X_2} = \pi^*(\Delta_0 + F)$.

Claim: (X_1, ω_1) and (X_2, ω_2) are not symplectomorphic.

In fact: $G_k(X_1, \omega_1) \not\simeq G_k(X_2, \omega_2)$.

One expects:

- $G_k^0(X_i, \omega_i)$ are solvable, $[G_k^0, G_k^0]$ is a quotient of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.
- $\text{Ab } G_k^0(X_1, \omega_1) \simeq (\mathbb{Z}_{3k})^{16k^2-1}$.
- $\text{Ab } G_k^0(X_2, \omega_2) \simeq (\mathbb{Z}_{4k})^{16k^2-1}$.

(Argument: refinement of Moishezon-Teicher techniques + A.H. perturbation techniques for iterated branched covers).