PROBLEM

Γ discrete group. What is the smallest n s.t. $\Gamma \underset{\text{properly}}{\curvearrowright} W^n \leftarrow$ contractible n-mfld ?

DEF. $\operatorname{actdim}(\Gamma)$ = this smallest n.

Ex. $\Gamma = F_2$, $\operatorname{actdim}(F_2) = 2$ gdim = 1

$\Gamma = F_2 \times F_2$ $\operatorname{actdim}(F_2 \times F_2) = 4$ gdim = 2.

DEF. $\operatorname{gdim}(\Gamma)$ = smallest n s.t.

$\Gamma \underset{\text{properly}}{\curvearrowright} P \leftarrow$ contractible n-polyhedron

Remark. Γ torsion-free

$\Rightarrow$ $\operatorname{gdim}(\Gamma) \leq \operatorname{actdim}(\Gamma) \leq 2 \cdot \operatorname{gdim}(\Gamma)$

Problem: Show $\operatorname{actdim}(F_2^n) = 2n$.

Idea

$\partial F_2 = C$ (= Cantor set)

$\partial(F_2^n) = \overset{n}{\underset{1}{*}} C$

If $F_2^n \curvearrowright W^{2n-1}$ ← think $\mathbb{R}^{2n-1}$

then should have

$$\partial F_2^n \subset \partial W^{2n-1}$$
$$\overset{n}{\underset{1}{*}} C \subset S^{2n-2}$$

But (van Kampen 1930's)

$$\overset{n}{\underset{1}{*}} C \supset \overset{n}{\underset{1}{*}} (3\text{ pts}) \not\hookrightarrow S^{2n-2}$$

DEF 1

A finite simplicial complex K is an m-obstructor complex if

1. $\exists$ collection $\Sigma = \{\sigma_i, \tau_i\}_{i=1}^{k}$ of unordered pairs of disjoint simplices that determines a $\mathbb{Z}_2$-cycle of dim m in $$\left(K \times K - \bigcup_{\sigma \in K} \sigma \times \sigma\right) / \mathbb{Z}_2.$$

2. $\exists$ generic map $K \xrightarrow{f} S^m$ s.t. $$\sum_{i=1}^{k} \#[f(\sigma_i) \cap f(\tau_i)]$$ is <u>odd</u>

3. $\forall \sigma^m \in K \quad \#\{v : \{\sigma, v\} \in \Sigma\}$ is <u>even</u>

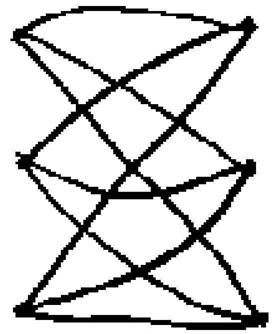
Perturb so $K \subset \mathbb{R}^{m+1}$

$$
\begin{tikzcd}
\tilde{\Sigma} \subset K \times K - \Delta \arrow[r] \arrow[d] \arrow[r, bend left, "\deg 1"] & S^m \arrow[d] \\
\Sigma \subset K \times K - \Delta / \mathbb{Z}_2 \arrow[r] \arrow[r, bend right, "\deg 1"'] & \mathbb{RP}^m
\end{tikzcd}
$$

Examples

1) utilities graph

2-obstructor



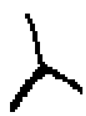
2) $S^n \cup pt = S^n_+$ n-obstructor

3) (join lemma)

K_i - n_i-obstructor $\Rightarrow$ $K_1 * K_2$ (n_1+n_2+2)-obst.

4) (cone lemma)

K n-obstructor $\Rightarrow$ cone(K) $(n+1)$-obstructor

 1-obstructor

Thm

1. (van Kampen)

K m-obstructor $\Rightarrow K \not\hookrightarrow \mathbb{R}^m$.

In fact, for every $f: K \to \mathbb{R}^m$ there are disjoint simplices $\sigma, \tau \in K$ s.t.

$$f(\sigma) \cap f(\tau) \neq \emptyset.$$

2. (linking lemma)

K m-obstructor, $F: K \times [0,\infty) \to \mathbb{R}^{m+1}$ proper map. Then $\exists$ disjoint $\sigma, \tau \in K$ s.t. $F(\sigma \times 0) \cap F(\tau \times [0,\infty)) \neq \emptyset$

- can replace $\mathbb{R}^m, \mathbb{R}^{m+1}$ by contractible m- and $(m+1)$- mflds.

Pf of the Linking Lemma

Let $H_t : \tilde{\Sigma} \longrightarrow S^m$

$H_t(x, y) =$ line from $F(x,0)$ to $F(y,t)$

Then $\deg H_0 = 1$

$\deg H_t = 0$ for $t \gg 0$. $\ast$

DEF 2

Γ virtually torsion-free discrete group

K finite simplicial complex

X proper metric space

We say

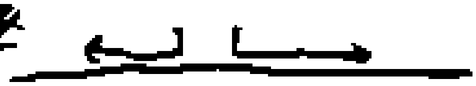
$$\text{“} K \subset \partial\Gamma \text{”}$$

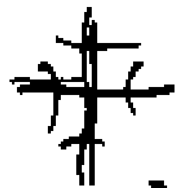
if

Γ acts isometrically, prop. disc., cocompactly on X and there is a proper expanding homotopy

$$H : K \times [0,\infty) \longrightarrow X$$

$\Big(\sigma, \tau \in K$ disjoint $\Rightarrow d\big(H(\sigma \times t), H(\tau \times s)\big) \to \infty$ as $s,t \to \infty.\Big)$

Ex. 1) $S^0 \subset \partial \mathbb{Z}$ 

2) $S^0_+ \subset \partial F_2$ 

3) $S^0_+ * S^0_+ \subset \partial(F_2 \times F_2)$, $\overset{n}{\underset{1}{*}} S^0_+ \subset \partial(F_2^n)$.

4) Γ hyperbolic, or CAT(0),

$K \subset \partial\Gamma \Rightarrow$ "$K \subset \partial\Gamma$"

DEF. obdim(Γ) = the ~~smallest~~ largest n s.t. "$K \subset \partial\Gamma$" for some $(n-2)$-obstructor K.

Ex. obdim$(F_2^n) = 2n$

THM 1 [B. - Kapovich-Kleiner]

m-obstructor complex

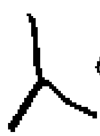
If "$K \subset \partial\Gamma$" then Γ cannot act properly discontinuously on a $(m+1)$-manifold.

i.e.

$$\boxed{\text{obdim}\ \Gamma \leq \text{actdim}\ \Gamma}$$

Ex 1) $\text{obdim}\ F_2^n = \text{actdim}\ F_2^n = 2n$

2) $BS(1,2) = \langle x, t \mid t^{-1}xt = x^2 \rangle$

"  $\subset \partial\Gamma$ "

$\text{obdim}\ \Gamma = 3$

$\text{actdim}\ \Gamma = 4$

Γ admits a unif. proper embedding to $\mathbb{R}^3$

3) M^n finite vol. hyperbolic n-mfld

Then "$S^{n-1}_+ \subset \partial \pi_1 M$"

so $\text{obdim}\, \pi_1 M = n$.

$\therefore$ $\text{obdim}\, \pi_1(M_1^{n_1} \times M_2^{n_2} \times \cdots \times M_k^{n_k}) = n_1 + n_2 + \cdots + n_k$

4) (Sung Yoon)

If $M_i^{n_i}$ is compact aspherical with aspherical incompressible ∂ components with inclusions of infinite index, then

$\text{actdim}\, \pi_1(M_1^{n_1} \times \cdots \times M_k^{n_k}) = n_1 + \cdots + n_k$.

Pf of Thm 1

$$K \times [0,\infty) \longrightarrow X \xrightarrow[\Gamma\text{-e.v.}]{} W^{m+1}$$
$$\cup$$
$$K \times [T,\infty)$$

apply the linking lemma to the composition

Thm 2 (B. — Feighn)

Γ lattice in a semisimple Lie group G.

$X = G/K$ associated symmetric space.

Then obdim Γ = act dim Γ = dim X.

Thm 3 (B. — Feighn)

M complete Riemannian contractible

G connected simple Lie gp, rank > 1, finite center

$\Gamma < G$ lattice

If dim $M <$ dim G/K then every isometric action $\Gamma \curvearrowright M$ must have a bounded orbit.

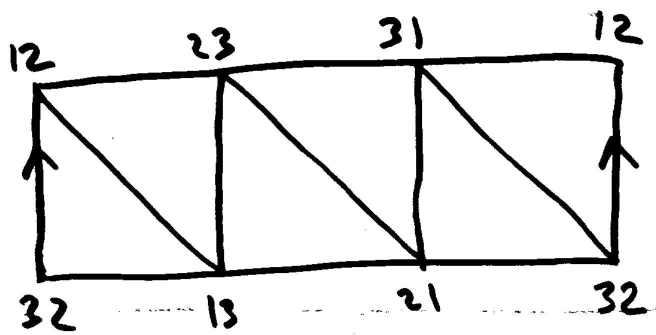
(false in rank 1)

$\underline{\underline{SL_3(\mathbb{Z})}}$ $\quad X = SL_3\mathbb{R}/SO_3$ 5-dim'l ⑪

$$B = \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \quad \text{Borel subgroup}$$

Consider: 6 conjugates of B

$$\begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & * & \\ & 1 & \\ * & * & 1 \end{pmatrix}, \begin{pmatrix} 1 & & * \\ * & 1 & * \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & * & * \\ & 1 & \\ & * & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ * & 1 & * \\ * & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ * & 1 & \\ * & * & 1 \end{pmatrix}$$



$= C$ "cuspidal complex"

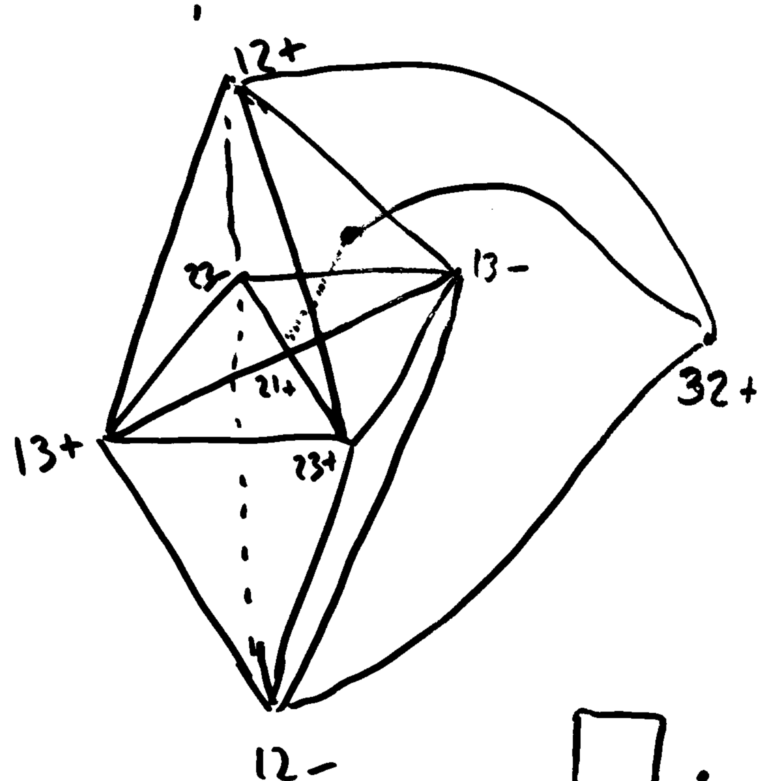
$$12 \leftrightarrow \begin{pmatrix} 1 & * & \\ & 1 & \\ & & 1 \end{pmatrix} \cong \mathbb{Z} \rightsquigarrow S^0 \subset \partial\Gamma$$

$$12 \text{——} 32 \leftrightarrow \begin{pmatrix} 1 & * & \\ & 1 & \\ & * & 1 \end{pmatrix} \cong \mathbb{Z}^2 \rightsquigarrow S^1 \subset \partial\Gamma$$

 $\leftrightarrow \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} = B \rightsquigarrow S^2 \subset \partial\Gamma$

There is a functorial procedure
sphericalization
that replaces each simplex Δ^n $\left(= \ast_1^{n+1}(pt)\right)$ by the
n-sphere S^n

$\ast_1^{n+1}(S^0)$



$= S^1_+ \ast S^0_+$

3-obstructor

$\therefore \underbrace{S^1_+ \ast S^0_+}_{} \subset \partial\, Sl_3\mathbb{Z}$

$\therefore$ obdim $Sl_3(\mathbb{Z}) = 5.$

$$\begin{pmatrix} 1 & & * \\ & 1 & * \\ & + & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & * & \\ + & 1 & \\ & & 1 \end{pmatrix}$$

S^1_+ $\qquad$ S^0_+ $+\mathbb{R}P$

<u>$SL_n(\mathbb{Z})$</u>

"$S^0_+ * S^1_+ * \cdots * S^{n-2}_+ \subset \partial SL_n\mathbb{Z}$"

$$\begin{pmatrix} 1 & & & * & \\ & 1 & & * & \\ & & \ddots & \vdots & \\ & & + & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix}$$

S^i_+

m-obstructor for

$m = 0 + 1 + \cdots + (n-2) + 2(n-2)$

$= \frac{n^2}{2} + \frac{n}{2} - 3$

$= \dim SL_n\mathbb{R}/SO_n - 2.$

$SL_n(\mathbb{Z}[\sqrt{2}])$:

$$G = SL_n\mathbb{R} \times SL_n\mathbb{R}$$

$$K = SO_n \times SO_n$$

$$a + b\sqrt{2} \longmapsto (a+b\sqrt{2},\, a-b\sqrt{2})$$

$$\text{“}S^{n-2} * S^1_+ * S^3_+ * \cdots * S^{2n-3}_+ \subset SL_n(\mathbb{Z}[\sqrt{2}])\text{”}$$

$$\underset{S^{2i-3}_+}{\begin{pmatrix} 1 & & * & \\ & 1 & * & \\ & & \ddots & \\ & & +1 & \ddots \\ & & & & 1 \end{pmatrix}} \quad \text{———} \quad \underset{S^{n-2}}{\begin{pmatrix} + & & 0 \\ & + & \\ & & \ddots \\ 0 & & & + \end{pmatrix}} \cong \mathbb{Z}^{n-1}$$

$$m = (n-3) + 1 + 3 + \cdots + (2n-3) + 2(n-1)$$

$$A = \begin{pmatrix} 1 & R & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & -R \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$A^{-1}B = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

Instead, replace B by

$$B' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -R \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -R & -R \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$A^{-1}B' = \begin{pmatrix} 1 & -R-1 & -1 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

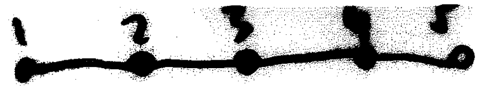
Key: Careful ordering of the simple roots.

$$\begin{pmatrix} 1 & \alpha_1 & & & \\ & 1 & \alpha_2 & & \\ & & 1 & \ddots & \\ & & & \ddots & \alpha_{n-1} \\ & & & & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & & \boxed{\ } & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

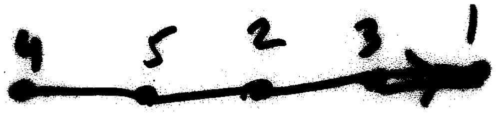
α_i + lower roots

$\Psi : \mathrm{Cone}(S^{n_1} * \cdots * S^{n_k}) \to G$

$\Psi(x_1, x_2, \ldots, x_k, d, y_1, \ldots, y_k) = \underbrace{x_1 x_2 \cdots x_k}_{\text{points in root spaces}} d \, \underbrace{y_1 y_2 \cdots y_k}_{\text{points on [illegible]}}$

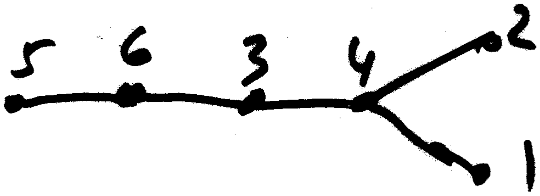


A_n

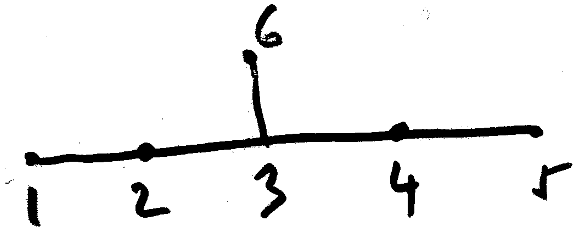


B_n

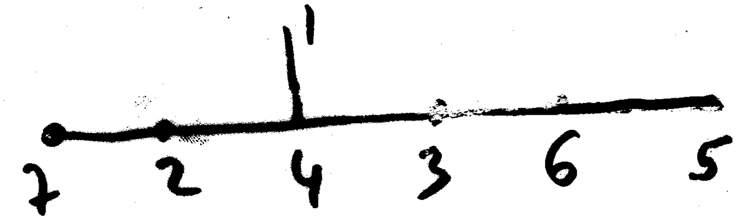
C_n



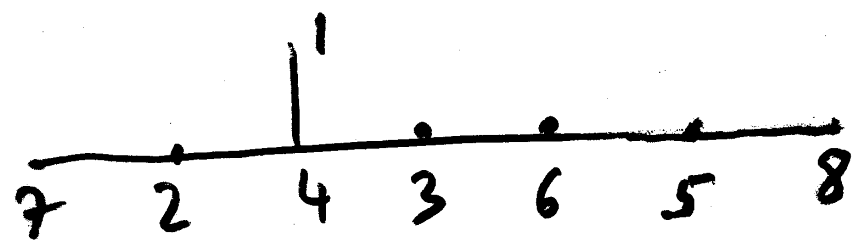
D_n



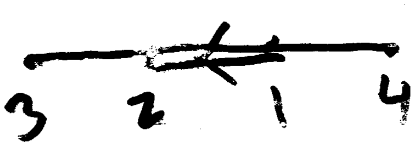
E_6



E_7



E_8



F_4