Representations of braid groups

Stephen Bigelow

bigelow@unimelb.edu.au

The Hecke algebra $H(q,n)$

A $\mathbf{C}$-algebra, where $q \in \mathbf{C}$, $n = 2, 3, \ldots$.

Generators: $g_1, \ldots, g_{n-1}$.

Relations:

- $g_i g_j = g_j g_i$ if $|i-j| > 1$,
- $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$,
- $(g_i - 1)(g_i + q) = 0$.

Assume $q \neq 0$, ($\Rightarrow g_i$ is invertible)

$\cdot$ q not a root of $\underline{1}$

($\Rightarrow H(q,n)$ is semi-simple)

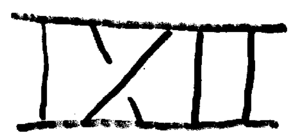
The Braid group B_n

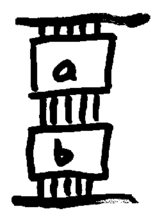
Generators: $\sigma_1, \ldots, \sigma_{n-1}$.

Relations:

- $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i-j| > 1$,
- $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$.

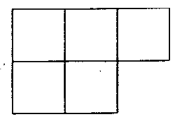
String:

eg. $\sigma_2 \in B_5$: 

$ab =$ 

Irreps of $H(q,n)$

If q is not a root of unity they correspond to partitions of n.



The restriction to $H(q,n-1)$ is the direct sum of all the ways to remove a corner box from the diagram.

Module M for a Young diagram

Number the boxes up-down, left-right.

M is the module $H(q,n)\mathbf{1}$ modulo

"1-block"

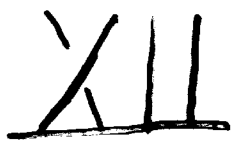
- $\sigma_i \mathbf{1} = -q\mathbf{1}$
  if $i, i+1$ are in the same column,

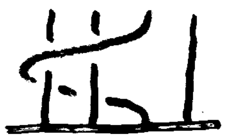
"2-block"

- $\sigma_j \ldots \sigma_i \sigma_i \ldots \sigma_j \mathbf{1} = q^{j-i}\mathbf{1}$
  if $i, \ldots, j$ is a column.

eg.

1	3	4
2		

 $= (-q)$ 

 $= q^1$ 

 $= q^0$ 

M as a $H(q, n-1)$-module

Say corner boxes are $c_1 < \ldots < c_k$

Let M_i be the $\mathbb{C}$-span of all:

string $\# j \geq c_i$

$M_1 \supset \cdots \supset M_k \supset M_{k+1} = 0$

Theorem

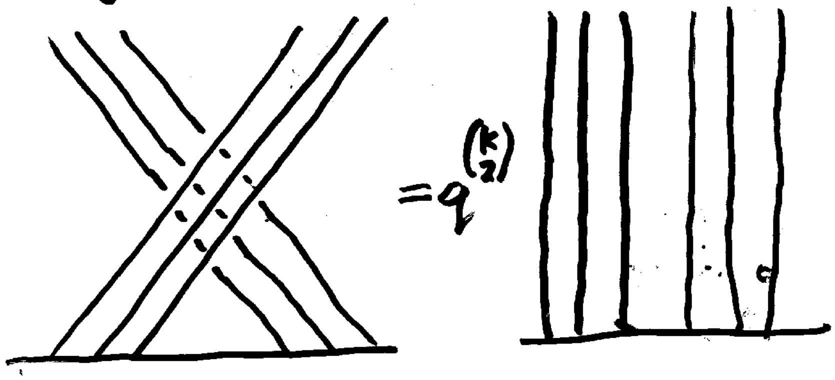
$$M = \bigoplus M_i / M_{i+1}$$

M_i / M_{i+1} = "remove box c_i"

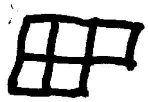
Lemma. $M \neq 0$. $M_i/M_{i+1} \neq 0$.

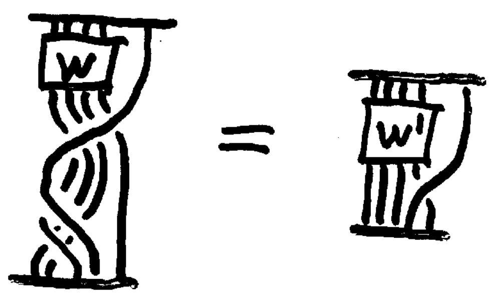
Lemma. For two columns of k boxes:

"Big X"



$\underline{M = M_1}$: Use $\sigma_i - q\sigma_i^{-1} = 1-q$ to pull top-right string in front. Then use 1-block, Big X to bring its bottom end to a c_j.

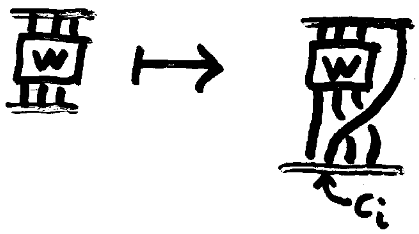
eg. for 



$$M = M_1/M_2 \oplus M_2 \quad \text{(semi-simple)}$$
$$= \bigoplus M_i/M_{i+1} \quad \text{(induct)}$$

$M_i' =$ "remove box c_i"

$\to M_i/M_{i+1}$



Well-defined: Check 1-block, 2-block of M_i' hold in M_i/M_{i+1}

Onto: Use 1-block, Big X to bring bottom end to a c_j as before

1-1: M' is irreducible, $M_i/M_{i+1} \neq 0$

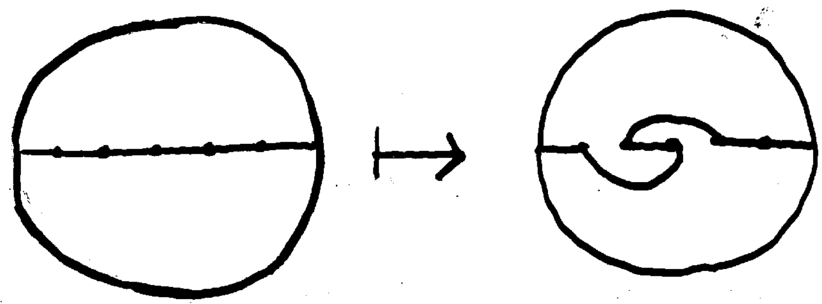
Mapping Class Group

$D_n = n$-punctured disk.

$B_n =$ MCG(D_n).

This is the group of homeos $D_n \to D_n$ which are id on ∂D_n, modulo isotopy.

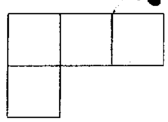
eg. $\sigma_2 \in B_5$:

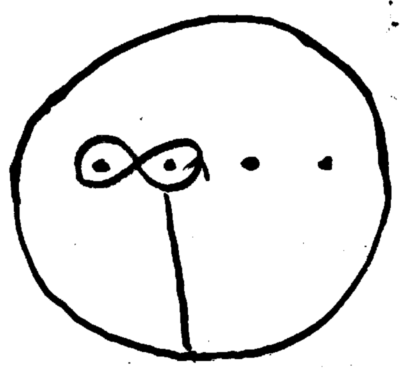


The Burau Representation

Let $\tilde{D}_n$ be the $\mathbf{Z}$-fold cover: wind counter-clockwise around a puncture – go up a level.

The Burau representation is the action of B_n on $H_1(\tilde{D}_n)$ by ~~$\mathbf{Z}$~~ $\mathbf{Z}[q^{\pm}]$-module homomorphisms.

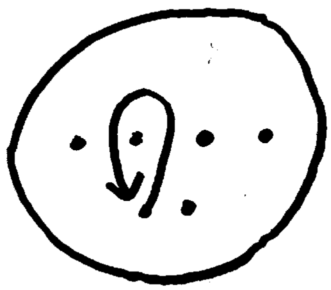
Compare 

$1 =$ 

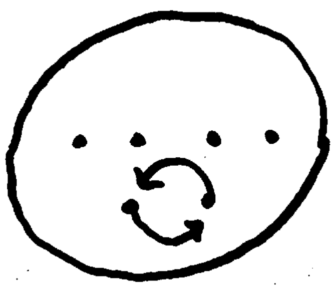
Lawrence-Krammer Representation

Let C be the space of unordered pairs of distinct points in D_n.

Let $\tilde{C}$ be the $\mathbf{Z} \oplus \mathbf{Z}$-fold cover:



$\mapsto q$



$\mapsto t$

The LK rep is the action of B_n on $H_2(\tilde{C})$.

Specializing

$H_2(\tilde{C})$ is a $\mathbb{Z}[q^{\pm1}, t^{\pm1}]$-module.

$\mathbb{C}$ is a $\mathbb{Z}[q^{\pm1}, t^{\pm1}]$-module

by $q \mapsto q_0$, $t \mapsto t_0$

$\mathbb{C} \otimes H_2(\tilde{C})$ is a $\mathbb{C}B_n$-module

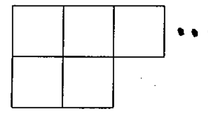
This is not the same as:

Find matrices for σ_i's,

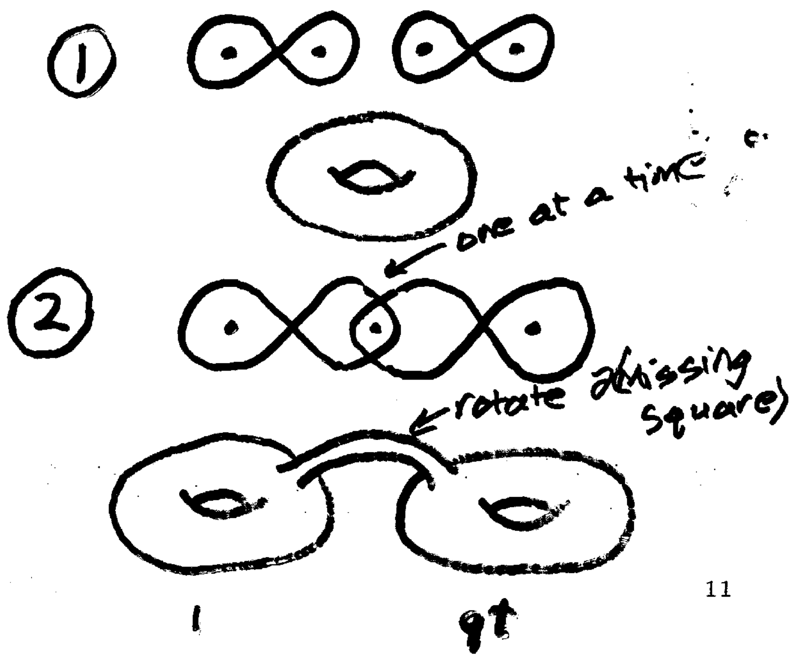
plug in values for q, t.

The latter may depend on your choice of basis for $H_2(\tilde{C})$, or rather for $\mathbb{Q}(q,t) \otimes H_2(\tilde{C})$

Theorem (Lawrence)

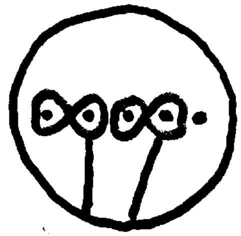
At $t = -q^{-1}$ this is reducible, and has  as a ~~sub rep~~. quotient

Sketch of Proof (me). Two types of surface:

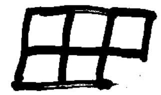


$V = \mathbb{C} \otimes H_2(\tilde{C})$

$V_0 =$ span of type 2 surfaces

$x =$  $\notin V_0$

consider $H_2(\hat{C}, N(\infty))$

$M = H(q,n)$-module for 
from first half of talk.

$\phi : M \to V/V_0, \quad 1 \mapsto x$

Well-defined: Check
1-block, 2-block, Hecke

1-1: M is irreducible

onto: icky explicit computation

THIS

GENERALISES

FINE PRINT:

One figure-eight for every column of two boxes.

Column of three: two red points and one orange. Orange pt. may coincide with punctures, but treats red pts like punctures. Red pts form type ② surface, while orange pt runs figure-eights around them.

Column of $\geqslant 4$: work in progress. Must work by Lawrence & first half of ~~talk~~

Possible Applications

- Faithfulness?
- Roots of unity!
- Jones polynomial?!