An introduction to geometric knot theory

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History

There are two main threads in this area:

1. Geometric properties of knots.

   Choose a geometric invariant of space curves, and minimize over a knot type.

   Crossing number (1880)
   # of quadrisecants (1930)
   Total curvature (1950)
   Distortion (1970)
   Renormalization energies (1990)
   Rope length (1990)

   Topology creates geometry.

2. Knot theories with restrictions.

Restrict attention to curves which obey additional geometric hypotheses, and study embedding classes.

Knots with positive curvature

Legendrian and transverse knots

Braid theory

Polygonal knots and Carpenter's Rule

Geometry creates topology.

The Distortion Problem

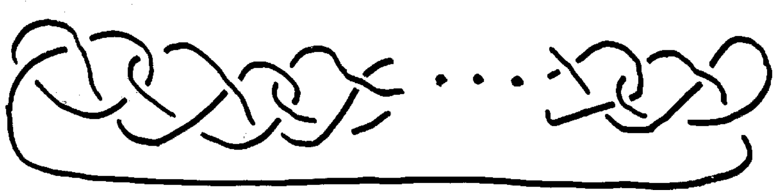
Defn. Given a space curve K the distortion of K is given by

$$\min_{p,q \text{ on } K} \frac{\text{distance from } p \text{ to } q \text{ in } \mathbb{R}^3}{\text{distance from } p \text{ to } q \text{ on } K}$$

Thm [Gromov] A closed curve has distortion at least $2/\pi$.

Can all Knots be constructed with distortion less than 100?

Thm [O'hara] $\exists$ prime knots of arbitrary crossing # with bounded distortion

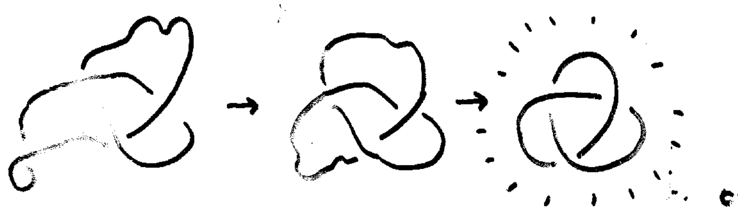


Renormalization Energies

Defn. The möbius energy of a curve K is given by

$$E[K] := \iint_{p,q \text{ on } K} \frac{1}{|p-q|^2} - \frac{1}{d(p,q)^2}\, ds\, dt$$

where $d(p,q)$ is the distance from p to q along the curve.



Thm [Freedman-He-Wang] There exists a minimizing curve in each prime knot type.

Thm [He] Any local minimizer for E is C^∞.

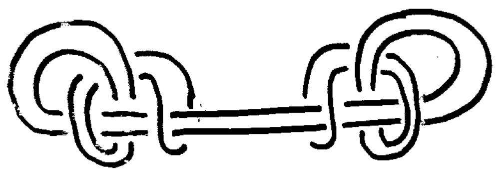
Thm [He] The gradient flow of E has a unique smooth solution for short time starting with any smooth loop.

Fundamental Questions on Knot Energies.

Can we untangle unknots by gradient flow?

-or-

Do there exist unknotted local minima for E?



?

Thm [He] There are no planar critical points for E (other than circle).

Thm [Freedman-He-Wang] The unique E-minimal unknot is the round circle.

Are these results true for other energy functionals or are they unique to E?

Defn. Given $F: \mathbb{R}^2 \to \mathbb{R}$, the renormalization energy based on F is the energy functional

$$F[K] := \iint_{p,q \text{ on } K} F(|p-q|, d(p,q))\, ds\, dt$$

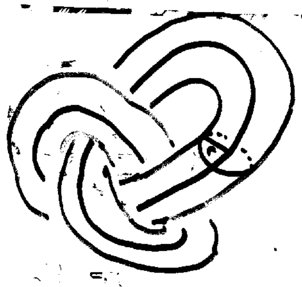
where $d(p,q)$ is the distance from p to q along K.

Thm [with Abrams, Fu, Ghomi, Howard] If $F(-,y)$ is decreasing and convex on $(0,y)$ for each y, then F is uniquely minimized by the round circle.

Work in Progress -or- Conjecture: If $F(-,y)$ is decreasing, then F has no non-convex planar critical points.

Ropelength

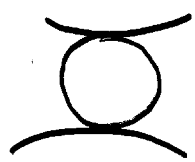
Defn. Given a C^1 curve, the thickness is the radius of the largest embedded ~~to~~ normal tube.



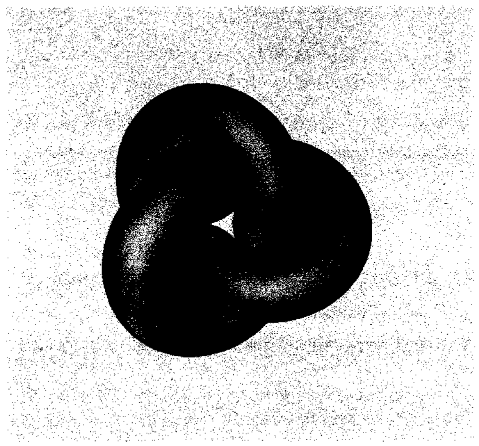
Thm [Gonzalez-Maddocks] Thickness is also given by the formula

$$T[K] := \min_{p,q,r \in K} R(p,q,r)$$

where $R(p,q,r)$ is the radius of the circle through p, q, and r.



Defn. The ropelength of a curve is given by length/thickness.



Thm [with Kusner, Sullivan] Ropelength minimizers exist and are $C^{1,1}$ in each knot and link type.

Find and describe minimizers!

Ropelength Minimizing Links

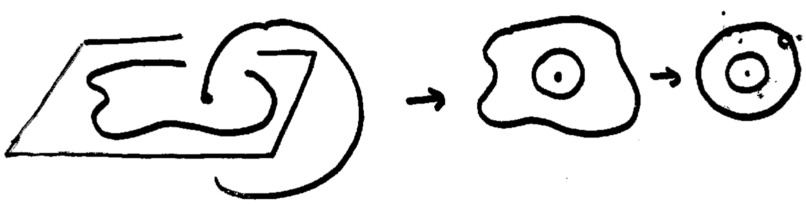
Intuition (Gehring link problem)

Given , want to minimize

length of  while staying ≥ 1 unit

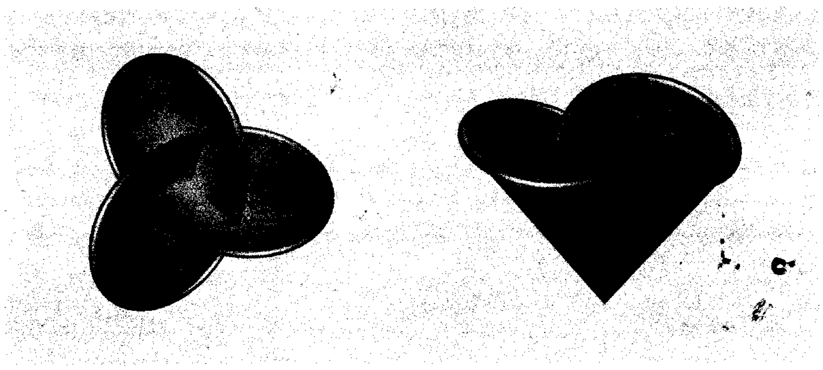
from .

If  was planar:



Goal: Find a way to reduce to planar case.

Thm [with Kusner, Sullivan] Every space curve bounds an intrinsically flat cone surface (a cone of cone angle 2π).



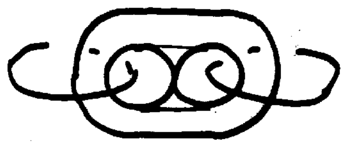
Thm [with Kusner, Sullivan] If K is linked to n other curves, then

$$\mathrm{Rop}(K) \geq \mathrm{Peri}(n) + 2\pi$$

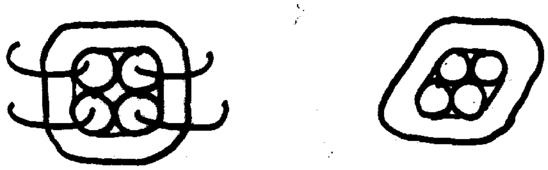
where Peri(n) is the length of the shortest curve around n disks of unit radius in the plane.

Consequences.

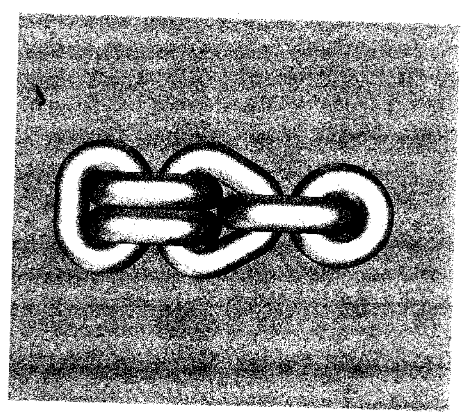
Ropelength minimizers are not C^2.



Ropelength minimizers are not unique.

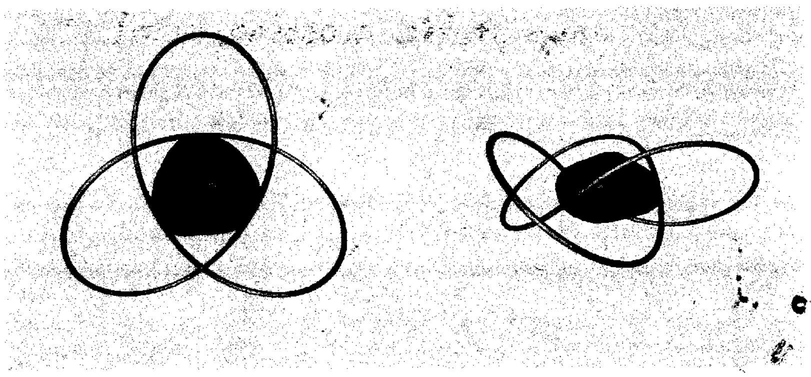


Can build a family of minimal links:



Can we find a better cone for knots?

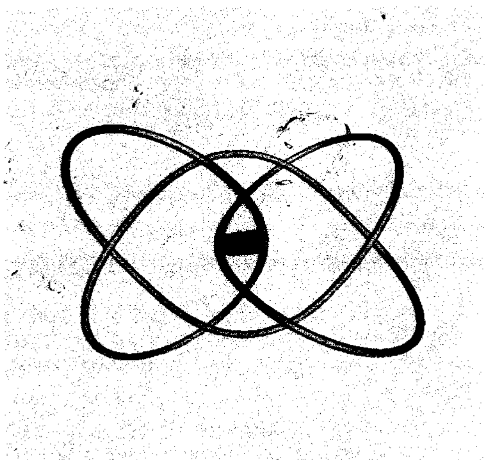
Defn. A point p is in the n-th hull of K if every plane through p cuts K in $2n$ points.



Fact: A point in the nth hull has cone angle at least $2\pi n$.

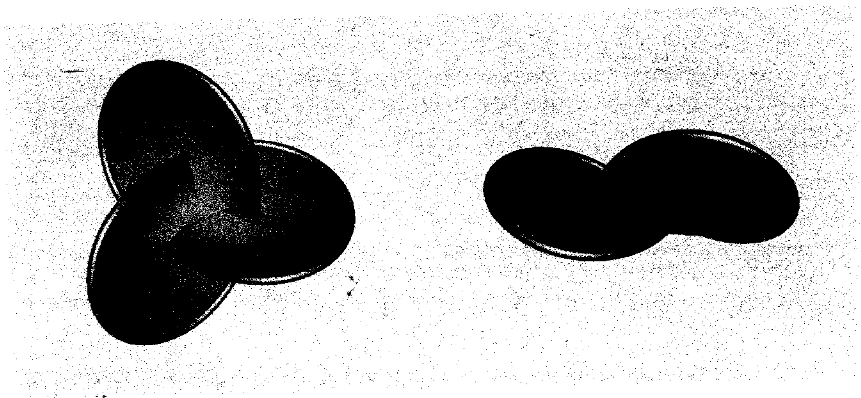
Thm [with Kusner, Sullivan, Kuperberg]
Every knot has a nonempty 2nd hull.

Is there a knot type for which every curve in K has nonempty nth hull (for n>2)?



This (3,4) torus knot is a prime 3-bridge knot. In this shape it has a 3rd hull.

Here is a 4π cone surface on a trefoil.



Defn. The overcrossing number of a link
A, B is the minimum number of times
A crosses over B in any projection.

Defn. The parallel overcrossing number $PC(K)$
is the min $Ov(K, K')$ where K' is any
parallel to K.

Thm [with Kusner, Sullivan] $PC(K) \geq$ bridge # (K).

Thm [with Kusner, Sullivan] For any nontrivial Knot, we have

$$\mathrm{Rop}(K) \geq 2\pi\sqrt{PC(K)} + 4\pi.$$

Defn. The asymptotic crossing number of a link L is given by

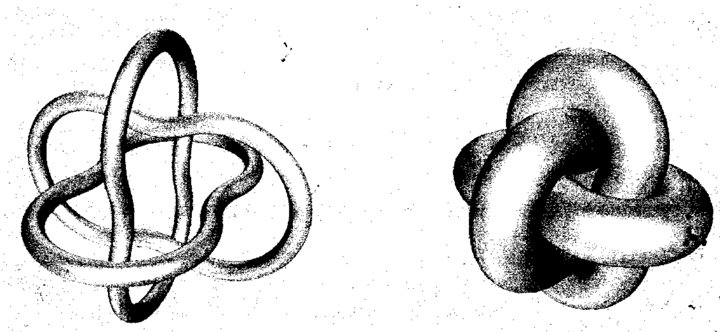
$$AC(L) = \inf_{pL, qL} \frac{Ov(pL, qL)}{|pq|}.$$

Fact: $Cr(L) \geq PC(L) \geq AC(L)$.

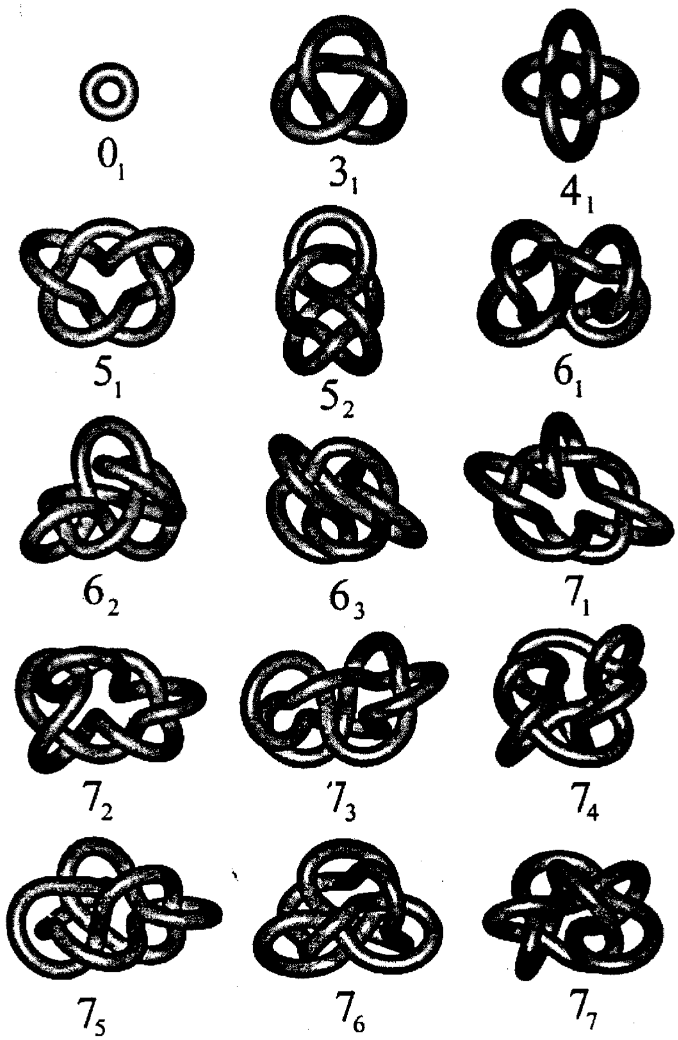
$$\boxed{\text{Is it true that } AC(L) \geq Cr(L)?}$$

"The mathematics of the 21st century"

What does it mean to be ropelength critical? Are there Euler-Lagrange equations for ropelength?



A conjectured ropelength-minimal Borromean rings.


0_1
3_1
4_1
5_1
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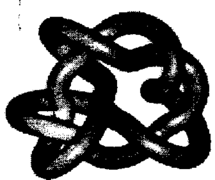

8_1

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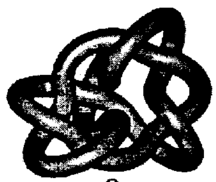

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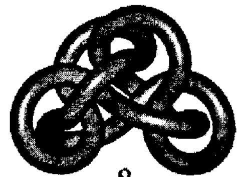

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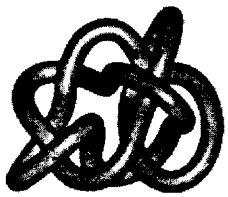

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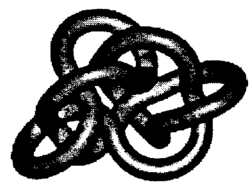

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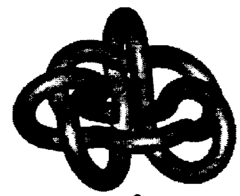

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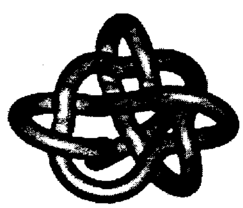
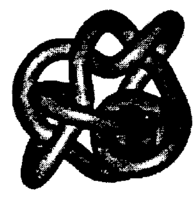


8_{10}



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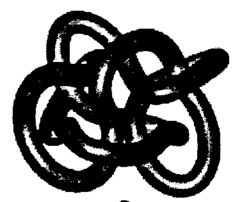

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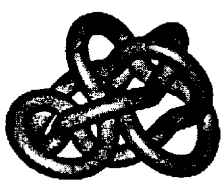
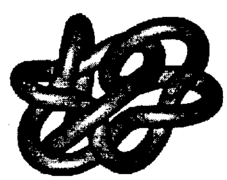


8_{13}



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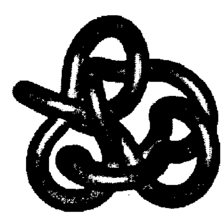

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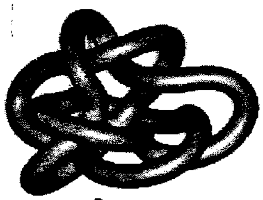
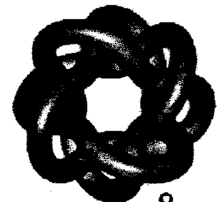


8_{16}

8_{17}



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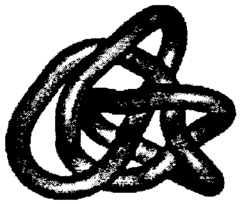
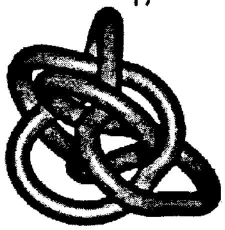


8_{19}



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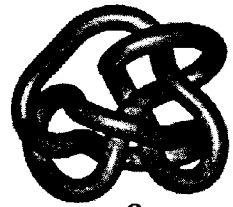
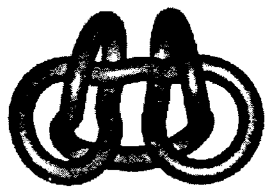
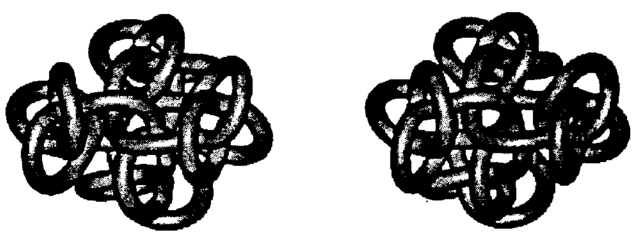


8_{21}

$3_1 \# 3_1$



$+3_1 \# -3_1$



$4_1 \ \# \ 4_1 \ \# \ 4_1 \ \# \ 4_1$