On the Topology of closed 3-manifolds with positive Scalar Curvature.

Joint work with
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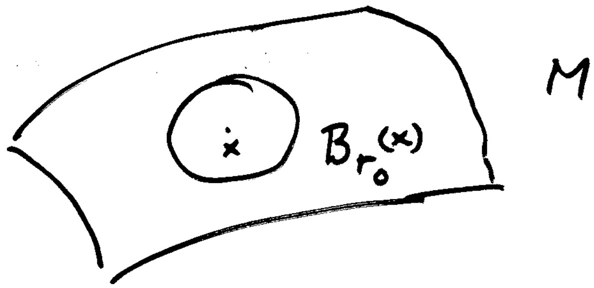
Let M^3 be a fixed (but arbitrary) closed 3-manifold and let g be a fixed (but arbitrary) integer.

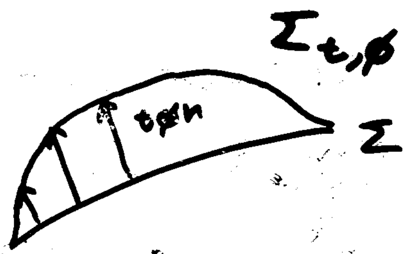
Question: Describe all closed embedded minimal surfaces in M with genus g?



Since sufficiently small balls in M^3 look roughly Euclidean the main problem for the local description is where:

$$\Sigma \subseteq B_{r_0} \subseteq \mathbb{R}^3 \leftarrow \text{with the flat metric}$$

is an embedded minimal surface with $\partial \Sigma \subseteq \partial B_{r_0}$



Variation of Σ

n unit normal

$\phi \in C_0^\infty(\Sigma)$

$$\Sigma_{t,\phi} = \{ x + t\phi(x) n(x) \mid x \in \Sigma \}$$

$$\frac{d}{dt}\Big|_{t=0} \text{Area}(\Sigma_{t,\phi}) = 0 \quad \text{for all } \phi \in C_0^\infty(\Sigma$$

$$\Longleftrightarrow$$

Σ is minimal

If Σ is minimal, then

$$\frac{d^2}{dt^2}\Big|_{t=0} \text{Area}(\Sigma_{t,\phi}) = -\int_\Sigma \phi L \phi$$

where

$$L\phi = \Delta_\Sigma \phi + |A|^2 \phi + Ric_M(n,n)\phi$$

Σ is said to be stable

$$\Longleftrightarrow$$

$$\frac{d^2}{dt^2}\Big|_{t=0} \text{Area}(\Sigma_{t,\phi}) \geq 0 \text{ for all } \phi \in C_0^\infty(\Sigma)$$

Here A is the second fundamental form. So

$$|A|^2 = k_1^2 + k_2^2$$

where k_i are the principal curvatures.

A metric on a closed 3-manifold 6.
is said to be <u>bumpy</u>

$\Longleftrightarrow$

<u>Area</u> is a <u>Morse function</u>

that is iff every critical point
is nondegenerate.

B. White: <u>Bumpy</u> <u>metric are generic</u>.

7.

The <u>Morse index</u> of a minimal
surface Σ is <u>#</u> of <u>negative</u>
<u>eigenvalues</u> of the
quadratic form

$$\frac{d^2}{dt^2}\Big|_{t=0} \text{Area}(\Sigma_{t,\phi}) = -\int_\Sigma \phi L \phi .$$

Example (**Plane**)

Example (**Helicoid**)

$\Sigma \subseteq \mathbb{R}^3$

$(s\cos t, s\sin t, t)$

$s, t \in \mathbb{R}$.

Multi-valued graphs

Definition (An N-valued graph).

An N-valued graph over an annulu $D_{r_2} \setminus D_{r_1}$ in the plane is a single valued graph over

$$\{ r_1 \le \rho \le r_2 , \ |\theta| \le N\pi \}$$

Here $\mathcal{P}$ is the universal cover of the punctured plane $\mathbb{C} \setminus \{0\}$ with global polar coordinates (ρ, θ).

Note that $\frac{1}{2}$ of the helicoid is a multi-valued graph.

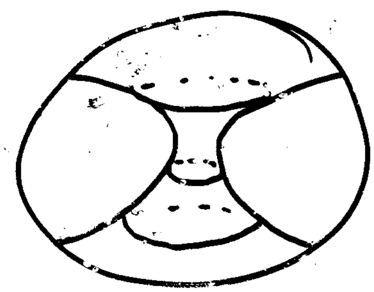
Here by $\frac{1}{2}$- of the helicoid we mean

$$(s\cos t, s\sin t, t)$$
$$s>0, \ t\in\mathbb{R}$$

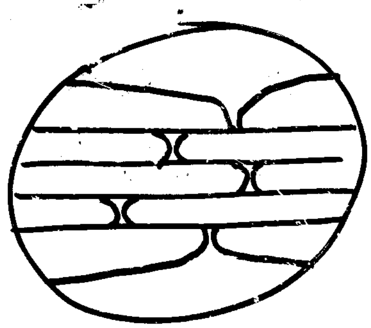
Example (Catenoid)

$\Sigma \subseteq \mathbb{R}^3$

$(\cos s \cosh t, \sin s \cosh t, t)$



Example (The Riemann Examples).



"Large curvature $\Rightarrow$ multi-valued minimal graph nearby"

<u>Thm</u> (C-M)

Given $\varepsilon > 0$ and $N > 1$, there exists $c_1, c_2 > 0$ such that:

If $\Sigma \subseteq B_{R_0}$ is an embedded minimal disk with $\partial\Sigma \subseteq \partial B_{R_0}$ and

$|A|^2(0) \geq c_1 r_0^{-2}$,

then $\underbrace{\Sigma}_{\text{(after rotation)}}$ contain an N-valued minimal graph Σ_g over $D_{R_0/c_2} \setminus D_{r_0}$ with gradient bounded by ε and $\Sigma_g \cap B_{4r_0} \neq \emptyset$.

13.

Blow-up Theorem:

"If the curvature of an embedded minimal disk is large at a point, then it contain a small multi-valued graph nearby. The size of ~~this~~ graph is on the scale of $\frac{1}{\sqrt{|\text{curvature}|}}$."

14.

Extension Theorem:

"A minimal disk ~~that~~ start off as a multivalued graph will remain so indefinite".

Convergence of embedded disks

Suppose that $\Sigma_i \subseteq B_{r_0} \subseteq \mathbb{R}^3$ is a sequence of embedded minimal disks with $\partial\Sigma_i \subseteq \partial B_{r_0}$.

Either the curvatures are bounded
or
The curvatures of a subsequence blow-up at some point

In either case by the above results there is a minimal disk through the point in the limit of a subsequence Σ_j.

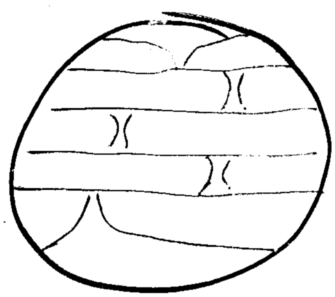
Estimate for stable minimal surfaces with small interior boundary.

Thm (C-M)

Suppose that $\Sigma \subseteq B_{r_0} \subseteq \mathbb{R}^3$ is an embedded stable minimal surface with $\partial\Sigma \subseteq \partial B_{r_0} \cup B_{\varepsilon r_0}$. Then away from the boundary Σ consists of (possibly several) components each of which is a graph over an annulus in the plane.

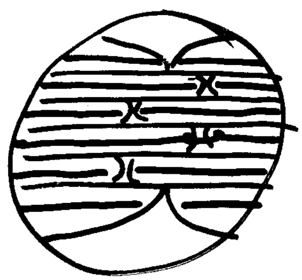
Pair of pants decomposition for minimal planar domains

Suppose that $\Sigma_i \subseteq B_{r_0} \subseteq \mathbb{R}^3$ is a sequence of embedded minimal planar domains with $\partial\Sigma_i \subseteq \partial B_{r_0}$. Either the intersection of each small extrinsic ball with each Σ_i consists of disks or (by the maximum principle) there are small noncontractible curves in a subsequence Σ_j



Σ_j

outline of decomposition



Σ_j

Γ stable

the case of the catenoid

outline: Solve a Plateau problem (using Meeks-Yau) to get a stable surface Γ. Show that Γ is a graph away from its boundary. Show that the original surface between two necks is itself a graph over Γ.

When we specialize our results to manifolds with has a metric with positive scalar curvature then we get:

<u>Thm</u> (C-M)

Let M^3 be a closed 3-manifold with a <u>bumpy</u> <u>metric</u> with <u>positive</u> <u>scalar curvature</u>. There exists at most <u>finitely</u> <u>many</u> <u>embedded minimal</u> surfaces in M of a given <u>fixed</u> <u>genus</u>.

Question (Pitts - Rubinstein)

Let M^3 be S^3 with a Riemannian metric. Is there a bound for the Morse index of all embedded minimal tori in M?

Pitts-Rubinstein: If this is the case for a sufficiently large class of metrics on S^3, then the spherical space-form problem can be solved affirmatively.

20.

Recall that the spherical space-form problem asks to show:

$$\underset{\text{universal cover}}{S^3 \longrightarrow M^3}$$

then M^3 admit a metric with constant sectional curvature one.

21.

Cor

If M^3 is a closed 3-manifold with a metric with positive scalar curvature, then the answer to the above question of Pitts-Rubinstein is Yes.