CONTACT TOPOLOGY PROBLEM SESSION. (GEORGIA TOPOLOGY CONFERENCE, MAY 26, 2001.)

The problem session was run by Emmanuel Giroux. We restricted attention to problems in 3-dimensional contact topology, that is, the more topological side of 3-dimensional contact geometry. There will also be another problem session on contact homology, symplectic field theory, and Legendrian curves.

Question 1 (Motivating Question)**.** *Can you find any application of contact geometry outside of contact geometry?*

Questions inside contact geometry:

1. Existence of contact structures (in dimension 3)

Question 2. *Which 3-manifolds have tight contact structures?*

Known: By Gabai, Eliashberg-Thurston, manifolds with non-zero b_1 admit universal tight contact structures.

More specifically, we have the following:

1. We need to prove existence on general Haken manifolds.
2. Which small Seifert fibered spaces have tight structures?
3. Distinctions: Universally tight, fillability.
4. Hyperbolic 3-manifolds: Try to construct tight structures.
5. $\pi_1(M)$ infinite acts properly discontinously on $\mathbf{R}^3$ (or $\mathbb{H}^3$) — can you make it act on $(\mathbf{R}^3, \xi_{std})$?
6. Construct taut foliations on hyperbolic M and perturb them into tight contact structures using Eliashberg-Thurston? (Rachel Roberts constructed a hyperbolic M w/o taut foliations, so clearly this cannot work in general.)
7. Find a good Dehn surgery theory for tight contact structures. Legendrian surgery can be done in a certain range on S^3, but there are other ranges where it isn't clear which surgeries should be tight).

2. Uniqueness/classification

Basically we have the same questions, but now we ask about uniqueness/classification.

Classification is known for: S^3, lens spaces, torus bundles, circle bundles.

1. Surface bundles over S^1 and Seifert fibered spaces. (Some of these are known. For example, $\#Tight(\Sigma(2,3,5)) = 0$ or 1, depending on orientation.)
2. Maybe Haken?

3. We would like to find a preferred contact structure on any manifold. Can one choose a distinguished contact structure?
4. Maybe there are only finitely many contact structures with minimal torsion? (Refers to toroidal 3-manifolds.)
5. Compute $\pi_0(Diff(M,\xi))$, where $Diff(M,\xi)$ stands for the space of contact diffeomorphisms. Study the map of this to $Diff(M)$.

   For example, this map is not surjective for T^3. It is for S^3. Perhaps the map is injective on π_0? On (T^3,ξ_n), π_0 has at least $\mathbf{Z}/n\mathbf{Z}$.
6. If you have $M_1\#M_2$, then $Tight(M_1\#M_2) = Tight(M_1)\times Tight(M_2)$. Is there a contact JSJ (Johannson-Shalen-Jaco) decomposition? Suppose $T_1, ..., T_k$ is a minimal system of tori given by the JSJ decomposition. Given classification on the pieces of $M \setminus (\cup_i T_i)$, can you give a classification?
7. Suppose there are two embeddings $(T^2\times I, \xi_n)\subset (M,\xi)$. If they are smoothly isotopic and maximal, then are they contact isotopic?
8. Is torsion always finite? (For example, if we drop the hypothesis that the torus is normal in Colin's theorem, does an arbitrary tight ξ exhibit this property? Note: For the infiniteness theorems of Colin and Honda-Kazez-Matić, the contact structures constructed are very special.)

3. Open books and Lefschetz fibrations

1. Fix g. Are there finitely many Heegaard splittings of genus g "compatible" with some tight contact structure? (Here, "compatible" means that it corresponds to a particular open book decomposition in the sense of Giroux.)
2. Fix M. Minimize g over all tight structures (or over all universally tight structures, etc.) on M such that corresponding Heegaard splitting is compatible. What is this? Can we omit tight here?
3. Calculate the global tb invariant of Giroux, and relate to the previous question.
4. Understand the monoid G_1 of $Map^+(\Sigma,\partial\Sigma)$ (orientation-preserving diffeomorphisms of Σ which fix $\partial\Sigma$) generated by the positive Dehn twists. Suppose Σ is a compact hyperbolic surface.

   We will now define a second monoid G_2. Consider universal cover $\widetilde{\Sigma}$ of Σ. $\widetilde{\Sigma}$ is a subset of $\mathbb{H}^2$. Pick a base point p on Σ as well as a lift $\tilde{p}$ of p on $\widetilde{\Sigma}$. Given $f\in Map^+(\Sigma,\partial\Sigma)$, pick a lift $\tilde{f}$ of f which fixes $\tilde{p}$. Now consider the action of $\tilde{f}$ on the circle of infinity S^1 of $\widetilde{\Sigma}$. This gives us an element of $Homeo^+(S^1)$, the orientation-preserving homeomorphisms of S^1. Viewing it as an element of $\widetilde{Homeo}^+(S^1)$, we can then ask whether or not $\tilde{f}(x)\geq x$. G_2 then consists of all non-decreasing f.

   $G_1\subset G_2$ by Ivan Smith. Are they equal?

   G_1 corresponds to Stein fillable. If G_2 is not the same as G_1, what does it correspond to? Symplectically fillable?

5. Is it possible to destabilize this monoid? Does destabilization preserve products of positive Dehn twists? Start with a fixed monodromy and stabilize. If it is then a product of positive Dehn twists, is the original map such a product?

   Positive evidence: Define a braid to be *quasi-positive* if it is a product of conjugates of positive generators. Then we have $B_n \to B_{n+1}$ (Stepan Orevkov) by mulitiplication by σ_n. Then destabiliation works.

   Given a surface $F \to D^2$ a simple branched cover (degree 3 or more), then any homeomorphism is isotopic to a lift of a braid of the branch points. If you stabilize the braid downstairs you stabilize the homeomorphisms. (If degree 4 or more) then every Dehn twist is covers a conjugate of a generator.
6. Find the kernel of the lifting map $F \to D^2$, i.e., liftable $L \subset B_n$. L surjects onto $Map(F, \partial F)$. What is the kernel? Find minimal set of generators (see Birman, Wainryb, and Montesinos-Morton).
7. Is every Leftshetz fibration a branched cover of a ruled surface? (This would follow from a good answer to the previous questions — squares and cubes.)
8. Can the contact homology be computed directly from an open book decomposition?

4. Fillings

1. Distinguish the various notions of filling below. The question mark "?" indicates we don't know whether the inclusions are strict inclusions.

Tight		
$\cup\ \not\vert$		
Weakly symplectically semi-fillable	$\supsetneqq$	Strongly symplectically semi-fillable
$\cup?$		$\cup?$
Weakly symplectically fillable	$\supsetneqq$	Strongly symplectically fillable
		$\cup?$
		Holomorphically fillable

2. Where does universal tightness fit into this picture?
3. What about exact symplectic filling?
4. Virtually filling question. Maybe some universally tight aren't fillable, but they become so after passing to a finite cover.
5. Fix (M, ξ). Does there exist a finite number of "minimal" fillings (X^4, ω) up to blow up, blow down, etc.?
6. How does one determine nonfillability?
7. Given M^4 described as a framed link surgery, does there exist a finite algorithm for computing the set of Stein structures.

5. Miscellaneous Problems

1. Understand virtually OT phenomenon.

   Picture of virtually OT phenomenon: Hopf-like unknot bounding an overtwisted disk, sometimes this can be unwound in a cover.

2. Compute the contact diffeomorphism group of (M, ξ) overtwisted?

   Does there exist an overtwisted contact structure which has 2 OT disks that are not contact isotopic? Eliashberg's theorem doesn't state that homotopy classes of plane fields are in 1-1 correspondence with overtwisted contact structures — this is only true at the π_0 level.

   (Note: there exist Legendrian knots in overtwisted structures with tight complements.)

3. Given an OT disk, can it always be completed to an OT tube? If so, is there an un-Lutz-twist to get tight structure?

6. Applications to 3-dimensional topology

1. Prove the Poincare conjecture.
2. Prove Geometrization and Riemann hypothesis.

For additional questions, please email `honda@math.uga.edu`.