Knots and Contact Geometry

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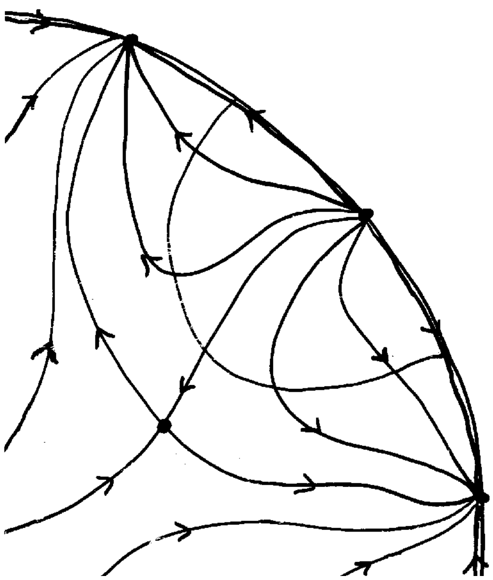


Plan of the Talk

I Background

II Motivation

III Main Results

IV Techniques

V Higher Dimensions

Background

Recall a hyperplane field ξ^{2n}

on a manifold M^{2n+1} is

called a <u>contact</u> <u>structure</u>

if there exists (at least locally)

a 1-form α such that

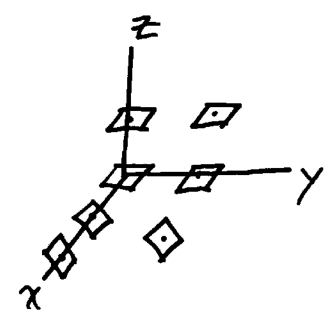
$$\xi = \ker\alpha \quad \text{and}$$

$$\alpha \wedge (d\alpha)^n \neq 0.$$

<u>example</u>:

1) on $\mathbb{R}^3$ consider $\alpha = dz + x\,dy$

$\xi_0 = \ker\alpha$



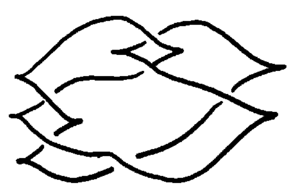
2) on $\mathbb{R}^{2n+1}$ consider $\alpha = dz + \sum_{i=1}^{n} x_i\, dy_i$

Consider the contact manifold
(M^{2n+1}, ξ^{2n}).

An embedded n-sphere L is called a <u>Legendrian knot</u> if $T_xL \subset \xi_x \quad \forall x \in L$

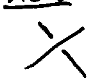
<u>example</u>:

If L is a Legendrian knot in $(\mathbb{R}^3, \xi_0)$, then its projection to the yz-plane "looks like"



key features : $<, >$ no vertical tangents

 but <u>not</u>



For most of the talk we restrict to the case $\boxed{n=1}$

There are two "classical" invariants of Legendrian knots

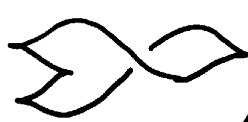
<u>Thurston-Bennequin Invariant</u>

$tb(L) =$ framing induced on normal bundle of L by ξ.

$=$ writhe$(L) - (\# \prec)$

in yz-projection

<u>example</u>:

tb () $= -1 + (-2)$

$= -3$

Rotation Number

Orient L by some $\vec{v} \in TL$

$r(L) =$ Euler number of $\xi|_\Sigma$ relative to $\vec{v}$, where Σ a surface, $\partial\Sigma = L$

$$= \tfrac{1}{2}(\# \prec \text{ or } \succ) - \tfrac{1}{2}(\# \prec \text{ or } \succ)$$

$\uparrow$

in yz-projection

<u>example</u>:

$r\Big($  $\Big) = 2$

Motivation

① Distinguishing Contact Structures

In the early 80's Bennequin showed

(1) $$\boxed{tb(L) + |r(L)| \leq -\chi(\Sigma)}$$

where $\partial\Sigma = L$ and L is a Legendrian knot in $(\mathbb{R}^3, \xi_0)$.

From this he concluded

Th^m: ξ_0 and

$\xi_1 = \ker(\cos r\, dz + r \sin r\, d\theta)$

are two distinct contact structures on $\mathbb{R}^3$.

Proof: consider

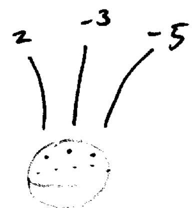
$L = \{r = \pi,\ z = 0\}$ in $(\mathbb{R}^3, \xi_1)$ ▦

def^n: call a contact structure tight if equation (1) holds for all Legendrian knots, otherwise call the contact structure overtwisted.

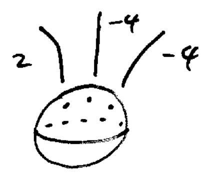
Colin, Giroux, Honda and Kanda have all used Legendrian knots to distinguish tight contact structures on various 3-manifolds

② Understand the Nature of Tight Contact Structures

$\underline{Th^{\underline{m}}}$ [E-Honda] If P is the Poincaré homology sphere, then $P \# \overline{P}$ does not admit any tight contact structure.

Recall $P =$ 

$\underline{Th^{\underline{m}}}$ [E-Honda] The manifold

$M =$ 

has a tight contact structure that is not symplectically fillable.

③ Relations with Braid and Knot Theory

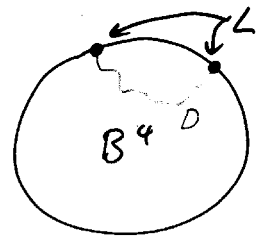
- Bennequin, Birman-Wrinkle and Menasco have found connections between braids and transverse knots (i.e. L s.t. $T_xL \pitchfork \xi_x$).
- Many people have found relations between tb, r and various knot polynomials.
- Giroux and Goodman can answer classical questions about fibered knots using contact geometry and transverse knot theory.

④ Understanding Knot Concordance

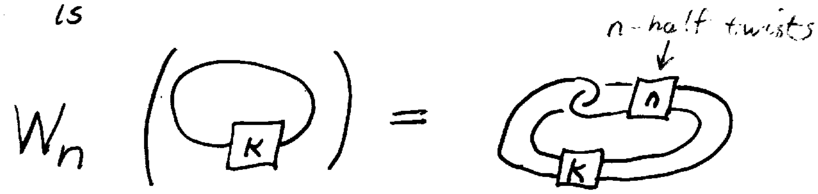
The trivial elements in the knot concordance group are slice knots

$L \subset S^3$ is slice if $\exists\, D^2 \subset B^4$ s.t.

$\partial D = L$



def$^{\underline{n}}$: the n-twisted double of a knot K is

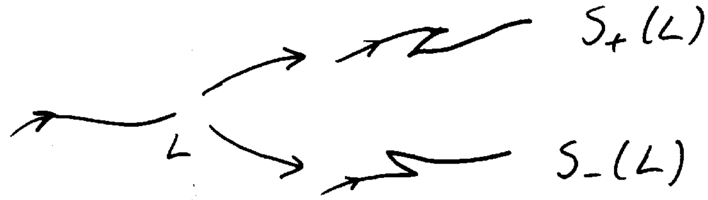


Th$^{\underline{m}}$: For any K there are only finitely many n for which $W_{2n}(K)$ is slice.

Main Results

Here we consider Legendrian knots in $(\mathbb{R}^3, \xi_0)$.

def$^{\underline{n}}$:



Stabilization

$$tb(S_\pm(L)) = tb(L) - 1$$
$$r(S_\pm(L)) = r(L) \pm 1$$

Th$^{\underline{m}}$ [Fuchs-Tabachnikov]

If L_1 and L_2 are topologically isotopic Legendrian knots with the same "classical invariants" then they are stably isotopic.

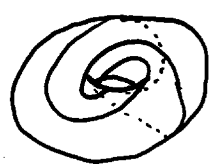
The first classification th$^{\underline{m}}$ was:

Th$^{\underline{m}}$ [Eliashberg-Fraser] Legendrian unknots are determined by their tb and r.

$$tb = -2s - t$$
$$r = \pm(t-1)$$

Th$^{\underline{m}}$ [E-Honda] Legendrian torus knots and figure eight knots are determined by their tb, r and topological knot type.

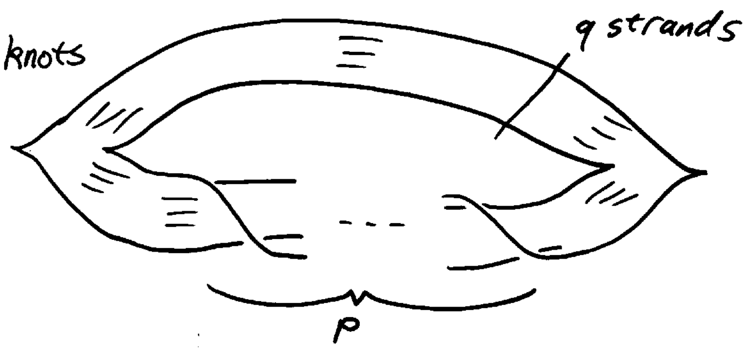
recall: torus knot   figure eight knot



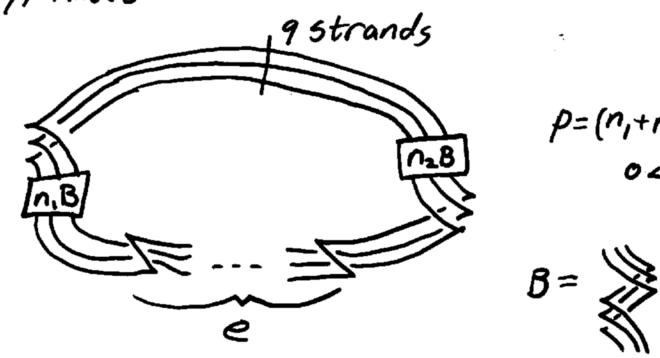
Knots with maximal tb

$p > q > 0$

(p,q)-knots



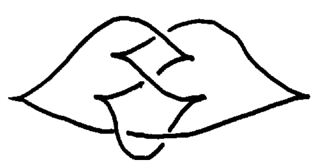
$(p,-q)$-knots



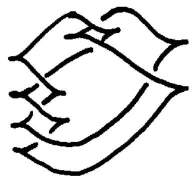
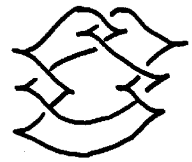
$p = (n_1 + n_2 + 1)q + e$

$0 < e < q$

figure eight



Not all Legendrian knots are determined by their tb and r.

e.g.  and 

are <u>not</u> Legendrian isotopic but both have $tb = 1$ and $r = 0$.

To distinguish these knots one must use <u>Contact</u> <u>Homology</u>.

(this was done by <u>Chekanov</u> and <u>Eliashberg</u>-<u>Hofer</u>)

(extensions by <u>E</u>-<u>Ng</u>-<u>Sabloff</u>)

Thm [Fuchs-Tabachnikov]

No finite type invariant can distinguish Legendrian knots with the same classical invariants.

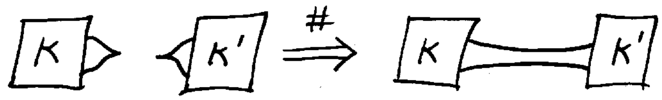
So Contact Homology and invariants extracted from it are infinite order.

Question: Can one extract a topological invariant of infinite order out of this?

more general results:

Th$^{\underline{m}}$ [E-Honda]

Maximal tb Legendrian knots have unique prime decompositions.



Applications:

1) Given any $n > 0$, there exist two topologically isotopic Legendrian knots with the same tb and r that do not become Legendrian isotopic untill they are stabilized atleast n times.

2) There are Legendrian knots in fibered knot types that are not determined by their classical invariants.

3) There are Legendrian knots that are <u>not Legendrian isotopic</u> but have the <u>same classical invariants</u> and all "<u>easily computable contact homology invariants</u>" are also the same.

Techniques of Proof

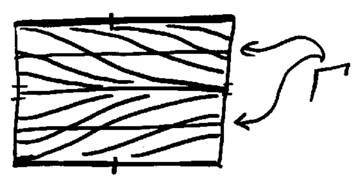
Convex Surfaces

let $\mathcal{F}$ be a singular foliation on a surface Σ

curves Γ divide $\mathcal{F}$ if

1) $\Gamma \pitchfork \mathcal{F}$
2) $\Sigma \setminus \Gamma = \Sigma_+ \sqcup \Sigma_-$
3) . . .

example: T^2



a surface Σ in (M, ξ) is convex if $\exists$ dividing curves for $\Sigma \cap \xi$.

Th$^{\underline{m}}$: Any closed surface is C^∞ close to a convex one. Also true if $\partial\Sigma \neq \emptyset$ as long as
1) $\partial \Sigma$ is Leg.
2) $tb(\partial\Sigma, \Sigma) \leq 0$

$\underline{\text{Th}^{\underline{m}}}$ [Giroux] if $\Sigma \cap \xi$ and $\mathcal{F}$ are divided by the same Γ then we can isotop Σ to Σ' so that $\Sigma' \cap \xi = \mathcal{F}$.

<u>Lemma:</u> $tb(\partial \Sigma, \Sigma) = -\frac{1}{2}(\Gamma \cap \partial \Sigma)$.

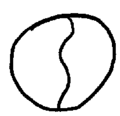
Proof of Legendrian Unknot Classification

Let L be a Legendrian Unknot. and D a disk it bounds.

<u>note</u>: $tb(L) \leq -\chi(D) = -1$

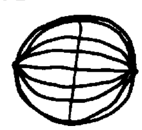
so we can make D <u>convex</u>.

I. if $tb(L) = -1$, then

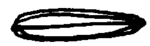


(cannot have a closed loop in Γ on D if ξ is tight)

so we can make

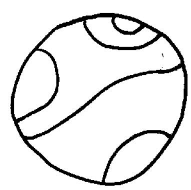
$\Sigma \cap \xi =$ 

then Legendrian isotop to

 (small neighborhood of a Legendrian arc)

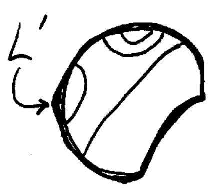
it is then easy to show any 2 such Legendrian knots are isotopic.

II. if $tb(L) < -1$, then



so we can make $D \cap \xi$:

define L' by



so $tb(L') = tb(L) + 1$

in fact: $L = S_{\pm}(L')$

Higher Dimensions

We now consider the contact structure $\xi = \ker\left(dz + \sum_{i=1}^{n} x_i\, dy_i\right)$ on $\mathbb{R}^{2n+1}$.

Question: Are there nontrivial Legendrian S^n's in $(\mathbb{R}^{2n+1}, \xi)$?

- if n is odd, then yes ($\exists$ classical invariants like tb).
- if n is even, then unknown till recently:

"Thm [Ekholm-E-Sullivan]" $\boxed{\text{Work in Progress}}$
$\exists$ nontrivial Legendrian S^n's in $(\mathbb{R}^{2n+1}, \xi)$ for all $n > 3$.

(should be ok for $n = 2, 3$ too, but...)

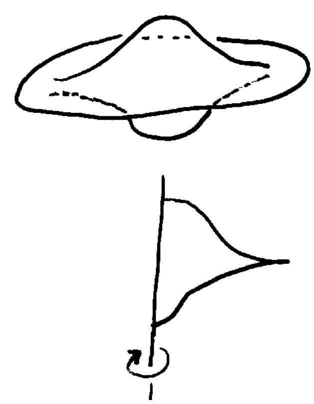
To understand Legendrian knots in $\mathbb{R}^{2n+1}$ we consider their "front projection"

$$\mathbb{R}^{2n+1} \longrightarrow \mathbb{R}^{n+1}$$
$$(x_1 y_1 \ldots x_n y_n\ z) \longmapsto (y_1 \ldots y_n\ z)$$

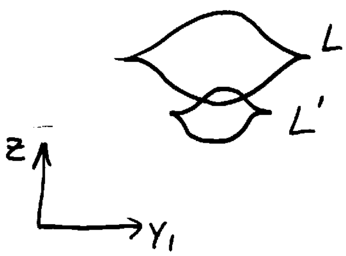
the projection can have complicated singularities, the simplest are "cusps"

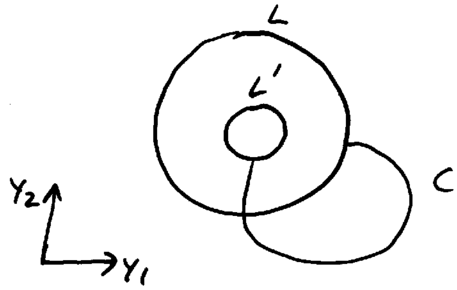
<u>example:</u> $n=1$

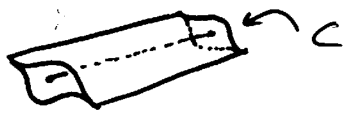
$n=2$

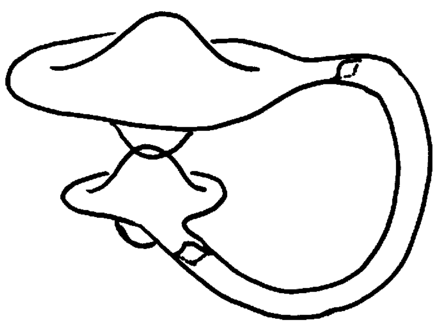


Our examples: (n=2)







$L_1 =$ 

$$L_k = L_{k-1} \# L_1$$

How do you distinguish the L_k's ?

Use <u>Contact</u> <u>Homology</u>

to define this consider the "complex" projection:

$$\mathbb{R}^{2n+1} \xrightarrow{\pi} \mathbb{R}^{2n}$$
$$(x_1 y_1 \cdots x_n y_n z) \longmapsto (x_1 y_1 \cdots x_n y_n)$$

let G be the set of double points of $\pi(L)$.

the chain groups $C(L)$ for the (<u>linear</u>) <u>contact</u> <u>homology</u> of L are the vector space (over, say, $\mathbb{Z}/2$) generated by G.

the boundary map $\partial : C(L) \to C(L)$ is defined by "counting holomorphic disks." (Similar to <u>Floer</u> <u>Homology</u>)

The (linear) contact homology of L is

$$\boxed{LCH(L) = \ker\partial / \operatorname{im}\partial}$$

There is a natural grading on $LCH(L)$

(coming from a Conley-Zehnder index)

<u>lemma:</u> $\dim(LCH_{-1}(L_k)) = k$ for $n > 3$.

(should be true for $n = 2, 3$ but need a little more work)