Volume, degree and entropy

Benson Farb
University of Chicago

Joint work with Chris Connell

The Degree Theorem

Degree Theorem. *Let M be a closed, locally symmetric n-manifold of nonpositive curvature. Assume that M has no local direct factors isometric to $\mathbb{R}^k, \mathbb{H}^2$, or $\mathrm{SL}_3(\mathbb{R})/\mathrm{SO}_3(\mathbb{R})$. Then for any closed Riemannian manifold N and continuous map $f : N \to M$,*

$$|\deg(f)| \leq C\frac{\mathrm{Vol}(N)}{\mathrm{Vol}(M)}$$

where C depends only on n and on the smallest Ricci curvatures of N and M.

Remarks

- Case when M negatively curved: Gromov, 1983.
- Case of surfaces M: follows from Gauss-Bonnet. (Plus)
- FALSE with $\mathbb{R}^k$ factors: the torus T^k has self-maps of any degree d.
- Dependence of C on Ricci curvatures is necessary.
- There is a finite volume version.
- **Entropy Rigidity Conjecture** (Katok) **(Gromov)**: give exact best constant C, with equality iff Riemannian covering.

  Proved for M:

  - negatively curved (Besson-Courtois-Gallot); settled many conjectures.
  - locally (not nec. globally!) isometric to a product of negatively curved (BCG, Connell-F).

Corollary: Positivity of Minvol

Minimal volume. One of the basic invariants of a smooth manifold is the *minimal volume* of N:

$$\mathrm{Minvol}(N) := \inf_g \{\mathrm{Vol}(N,g) : |K(g)| \leq 1\}$$

Corollary 1. [Positivity of Minvol] *M as in degree theorem* $\Longrightarrow \mathrm{Minvol}(M) > 0$.

Compact case: proved by Savage (1982) for $M = \Gamma \backslash \mathrm{SL}_n(\mathbb{R}) / \mathrm{SO}_n(\mathbb{R})$ and by Gromov (1983) in general.

Open question: Compute $\mathrm{Minvol}(M)$. Is it realized by the locally symmetric metric?

YES (even uniquely!) for real-hyperbolic M (B-C-G).

(n=2 follows from Gauss-Bonnet)

$K \geq -1$ $\quad \int_\Sigma K\,dA = 2\pi\chi(\Sigma)$

Corollary: Self-maps

Corollary 2. [No self-maps] *M as in degree theorem $\Longrightarrow$ M admits no self-maps of degree > 1.*

In particular, $\pi_1(M)$ is co-Hopfian*: every injective endomorphism of $\pi_1(M)$ is surjective.*

Also follows from Margulis Superrigidity. Co-Hopf property first proved by Prasad (1976).

Proof of Corollary 2: $\deg(f^n) = \deg(f)^n$. Now apply the Degree Theorem. $\diamond$

Proof of the Degree Theorem

Main Theorem: Any $f : N \to M$ is homotopic to a smooth map F with $\operatorname{Jac} F \leq C$.

- Unlike harmonic maps, $D_x F$ is explicit.
- This implies the Degree Theorem since:

  degree theory $\Longrightarrow$

$$
\begin{aligned}
|\deg(f)|\,\mathrm{Vol}(M) &= \text{[redacted]}\ |\deg F|\,\mathrm{Vol}\,M \\
&= \left|\textstyle\int_N F^* dg_M\right| \\
&\leq \textstyle\int_N |\operatorname{Jac} F| dg_N \\
&\leq C\,\mathrm{Vol}(N)
\end{aligned}
$$

Constructing the remarkable map F

(after Douady-Earle and B-C-G)

Lift $f : N \to M$ to $\widetilde{f} : Y \to X$. Today: assume $K(Y) \leq 0$ (VERY SPECIAL!)

To construct $\widetilde{F} : Y \to X$ we:

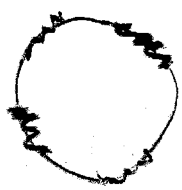
Step 1: Find a natural family of probability measures ν_y **parameterized by** $y \in Y$.

Step 2: Find a natural map $\partial\widetilde{f} : \partial_\infty Y \to \partial_\infty X$.

Step 3: Take the *barycenter* of a probability measure on $\partial_\infty X$, giving a map $\mathrm{bar} : \mathcal{M}_1(\partial_\infty X) \to X$.

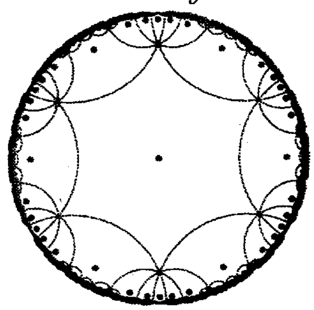
Then define $\widetilde{F}$ by:

$$\begin{tikzcd}
\mathcal{M}_1(\partial_\infty Y) \arrow[r, "(\partial\widetilde{f})_*"] & \mathcal{M}_1(\partial_\infty X) \arrow[d, "\mathrm{bar}"] \\
Y \arrow[u] \arrow[r, dashed, "\widetilde{F}"] & X
\end{tikzcd}$$



Step 1: Patterson-Sullivan Measures

Density of $\Gamma \cdot p$ at infinity as viewed from $y \in Y$ gives Patterson-Sullivan measure ν_y on $\partial_\infty Y$.



Key Properties of Patterson-Sullivan Measures ν_x:

1. Equivariance: $\nu_{\gamma x} = \gamma_* \nu_x$

2. Explicit Radon-Nikodym derivative $\frac{d\mu_y}{d\mu_x}$

   closure of F

3. (Knieper, Albuquerque) Each ν_x is supported on a specific Γ-orbit in $\partial_\infty Y$.

Step 2: Constructing the map
$\partial\widetilde{f} : \partial_\infty Y \to \partial_\infty X$

Morally:

1. By nonpositive curvature, Y and X have "visual compactifications" $\partial_\infty Y$ and $\partial_\infty X$.

2. Extend $\widetilde{f} : Y \to X$ to these compactifications using coarse geometry (where do geodesic rays go?).

<u>Warning</u>: can't <u>really</u> do this!

Step 3: Barycenter Functional

Busemann functions: Pick basepoint $p \in X$. For $x \in X$, $\theta \in \partial_\infty X$,

$$B(x,\theta) = \lim_{z\to\theta} d(x,z) - d(p,z)$$

Convex Functional:

$$\mathcal{B} : X \times \mathcal{M}_1(\partial_\infty X) \to \mathbb{R}$$

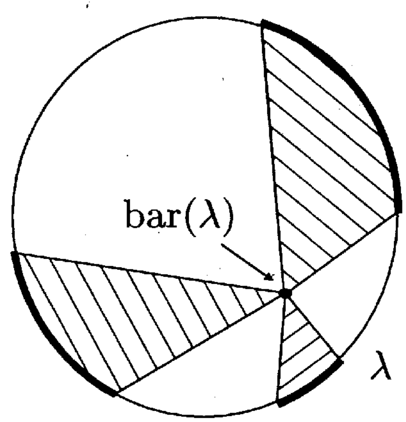
$$\mathcal{B}(x,\lambda) = \int_{\partial_\infty X} B(x,\theta)\, d\lambda(\theta)$$

Key fact: $\mathcal{B}(\cdot,\lambda)$ strictly convex.

Step 3: Barycenter map and F

$$\mathrm{bar} : \mathcal{M}_1(\partial_\infty X) \to X$$

$$\mathrm{bar}(\lambda) := \text{unique critical point of } \mathcal{B}(\cdot, \lambda)$$



Now define $\widetilde{F}$ by

$$\widetilde{F}(x) := \mathrm{bar}((\partial \widetilde{f})_* \nu_x))$$

Estimating $|\,\mathrm{Jac}\, F|$

Let $\sigma_y = (\partial\widetilde{f})_*\nu_y$. Now F is defined by the implicit vector equation:

$$\int_{\partial_\infty X} dB_{(F(y),\theta)}(\cdot)d\sigma_y(\theta) = 0$$

Differentiating, taking det, making estimates, rearranging, etc. gives:

$$\mathrm{Jac}\, F \leq C \frac{\det\left(\int_{\partial_\infty X} dB(\cdot)^2 d\sigma_y(\theta)\right)^{1/2}}{\det\left(\int_{\partial_\infty X} DdB(\cdot,\cdot)d\sigma_y(\theta)\right)}$$

Key fact: Can replace $\partial_\infty X$ by maximal compact $K \subset G = \mathrm{Isom}(X)$.

Same estimate as B-C-G.
In hyperbolic cases, a miracle occurs: can write bottom in terms of top.

Convert to a Lie groups problem

Now combine:

- Eigenvalue estimates on DdB
- For M_i positive semidefinite, $\det(\sum M_i)$ is a nondecreasing homogeneous polynomial in eigenvalues of the M_i.

to obtain

$$|\operatorname{Jac} F| \leq C \frac{\left(\det \int_K O_\theta \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} O_\theta^* \, d\sigma_y(\theta)\right)^{\frac{1}{2}}}{\det \int_K O_\theta \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} O_\theta^* \, d\sigma_y(\theta)}$$

where I is the identity matrix of size $n - \operatorname{rank}(X)$ and O_θ is an element of K.

It remains to bound the right hand ratio.

Eigenvalue Matching

Idea: For each small eigenvalue in the denominator, find two comparably small eigenvalues in the numerator.

Hard part: Make "comparably small" independent of the measure $(\partial\widetilde{f})_*\nu_y$.

Involves: Detailed analysis of action of maximal compact K on subspaces of Lie algebra of $G = \mathrm{Isom}(X)$ (a simple Lie group). Use:

- kernel (numerator) $\supseteq$ cokernel (denominator).
- must find K-invariant subspace perpindicular to K-orbit of a flat, and of TWICE dimension of flat ($=\mathrm{rank}(X)$). THIS USES $G \neq \mathrm{SL}(3, \mathbb{R})$!

Open problems

1. Entropy Rigidity Conjecture

2. Computing Minvol

3. Degree theorems for manifolds which are not locally symmetric. [For nonpositively curved manifolds with negative Ricci curvature?]

4. Real convex projective structures on surfaces

5. Homotopy Hyperbolic Theorem

6. Positivity of Gromov Norm for nonpos. curved, locally sym mnflds with no torus factors.