SO(3) MONOPOLES AND RELATIONS BETWEEN 4-MANIFOLD INVARIANTS

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THE PROBLEM ...

X IS A CLOSED,
ORIENTED SMOOTH
4-MANIFOLD, $b_2^+ \geq 1$

- DONALDSON INVARIANTS
DEFINED USING SO(3)
GAUGE THEORY
(YANG-MILLS)

- SEIBERG-WITTEN (SW)
INVARIANTS
DEFINED VIA U(1) GAUGE
THEORY (MONOPOLES)

PROBLEM PROVE WITTEN'S
CONJECTURE RELATING
D AND SW INVARIANTS

REMARKS ON THE PROBLEM

- CLASSICAL VS QUANTUM FIELD THEORY ?
  - WITTEN'S ARGUMENT USES "N=2 SUSY QUANTUM YANG-MILLS THEORY" AND "DUALITY"
  - METHOD WE USE INVOLVES ONLY CLASSICAL GAUGE FIELD THEORY. (SO(3) MONOPOLES)
- RESCALING METRIC g ON X?
  - METRIC RESCALING PLAYS KEY ROLE IN WITTEN'S APPROACH.
  - IN CLASSICAL, SO(3) MONOPOLE APPROACH, ALL GAUGE THEORIES SCALE INVARIANT.

SPINORS

- RIEMANNIAN METRIC g ON X

$$\begin{tikzcd} V \arrow[d] \\ X \end{tikzcd}$$

HERMITIAN VECTOR BUNDLE, WITH CLIFFORD MAP,

$$\rho : T^*X \longrightarrow \mathrm{END}_{\mathbb{C}}(V)$$

OBEYING:

$$\rho(\alpha)^\dagger = -\rho(\alpha), \qquad \alpha \in \Omega^1(X,\mathbb{R})$$

$$\rho(\alpha)^2 = -g(\alpha,\alpha)$$

COMPLEX CLIFFORD-ALGEBRA MODULE: (ρ, V)

- $s = (\rho, W)$ IS A SPIN^c STRUCTURE WHEN RANK$_{\mathbb{C}} W = 4$
- $t = (\rho, V)$ IS A "SPIN^u" STRUCTURE WHEN RANK$_{\mathbb{C}} V = 8$.

  GIVEN ANY W, $V \cong W \otimes_{\mathbb{C}} E$ FOR SOME $E \to X$, RANK$_{\mathbb{C}} E = 2$.

SPIN^u STRUCTURES AND GAUGE THEORIES

USE SPIN^u STRUCTURES TO PROVIDE COMMON FRAMEWORK FOR

- SO(3) YANG-MILLS THEORY
- U(1) MONOPOLES
- SO(3) MONOPOLES (CONTAINS PRECEDING TWO THEORIES)

$$su(V) = \{M \in \mathrm{END}_{\mathbb{C}}(V) : M^{\dagger} = -M \text{ AND } \mathrm{TR}(M) = 0\}$$

$$su(V^{\pm}) \cong su(W^{\pm}) \oplus i\,su(W^{\pm}) \otimes su(E) \oplus su(E)$$

$V = V^{+} \oplus V^{-}$, $W = W^{+} \oplus W^{-}$, ∓ 1 EIGENSPACES OF $\rho(\mathrm{VOL})$

- $\det^{\frac{1}{2}}(V^{+}) = \det(W^{+}) \otimes \det(E)$
- $\mathfrak{g}_{t} = su(E)$

CANONICALLY ASSOCIATED TO $t = (\rho, V)$.

SO(3) or PU(2)-monopole equations

Introduced by Pidstrigatch and Tyurin (December 1994), with cobordism idea to prove Witten's conjecture. A pair (A, Φ) is a *PU(2) monopole* if

$$\mathfrak{S}(A, \Phi) = \begin{pmatrix} \mathrm{ad}^{-1}(F_{\hat{A}}^{+}) - \rho^{-1}(\Phi \otimes \Phi^*)_{00} \\ D_A \Phi \end{pmatrix} = 0,$$

where

- A is a spin connection on V, with $A|_{\det(V^+)} = 2A_\Lambda$,
- $\hat{A}$ is induced connection on SO(3) bundle $\mathfrak{g}_{\mathfrak{t}} \subset \mathfrak{su}(V)$,
- $(\Phi \otimes \Phi^*)_{00} \in C^\infty(\rho(\Lambda^+) \otimes \mathfrak{g}_{\mathfrak{t}})$,
- $\rho : \Lambda^+ \cong \mathfrak{su}(W^+)$,
- $D_A : C^\infty(V^+) \to C^\infty(V^-)$ is Dirac operator,
- $\Phi \in C^\infty(V^+)$, where $V = V^+ \oplus V^-$.

Moduli space of PU(2) *monopoles* for $\mathfrak{t} = (\rho, V)$:

$$\mathcal{M}_{\mathfrak{t}} = \{(A, \Phi) : \mathfrak{S}(A, \Phi) = 0\}/\mathcal{G}_{\mathfrak{t}},$$

where $\mathcal{G}_{\mathfrak{t}} \cong \mathcal{G}_E$ and

$$\mathcal{G}_{\mathfrak{t}} = \{u \in \mathrm{Aut}_{C\ell}(V) : u^\dagger u = 1 \ \& \ \det_{C\ell}(u) = 1\},$$

$$\mathcal{G}_E = \{v \in \mathrm{Aut}(E) : v^\dagger v = 1 \ \& \ \det(v) = 1.$$

Singularities in PU(2)-monopole space

If $[A, \Phi]$ is a fixed-point in PU(2) monopole space, $\mathcal{M}_{\mathfrak{t}}$, under S^1 action given by scalar multiplication on V, then:

- (A, Φ) is an ***anti-self-dual*** solution:

$\Phi \equiv 0$ and $\hat{A}$ obeys $F^+_{\hat{A}} = 0$, so

$$M^w_\kappa = \{\hat{A} : F^+_{\hat{A}} = 0\}/\mathcal{G}_E \hookrightarrow \mathcal{M}_{\mathfrak{t}},$$

where $\kappa = -\frac{1}{4}p_1(\mathfrak{t})$ and w is any integral lift of $w_2(\mathfrak{t})$.

$:= p_1(su(E))$ $\quad$ $:= w_2(su(E))$

- **(A, Φ) is a *Seiberg-Witten*** or *reducible* solution:

$A = B \oplus B \otimes A_L$ when $V = W \oplus W \otimes L$, some line bundle L, and $\Phi = \Psi \oplus 0$ with $\Psi \in C^\infty(W^+)$, so

$$\mathfrak{S}(\overset{B,\Psi}{\text{[illegible]}}) = \begin{pmatrix} \underset{\text{Tr}}{(F^+_B)} - \rho^{-1}(\Psi \otimes \Psi^*)_0 - F^+_{A_\Lambda} \\ D_B \Psi \end{pmatrix} = 0.$$

Then $[B, \Psi] \in M_{\mathfrak{s}}$, where $\mathfrak{s} = (\rho, W)$ and

$$M_{\mathfrak{s}} = \{(B, \Psi) : \mathfrak{S}(B, \Psi) = 0\}/\mathcal{G}_{\mathfrak{s}} \hookrightarrow \mathcal{M}_{\mathfrak{t}},$$

where $A_\Lambda = B^{\det} \otimes A_L$ and

$$\mathcal{G}_{\mathfrak{s}} = \{s \in \operatorname{Aut}_{C\ell}(W) : s^\dagger s = 1\} \cong \operatorname{Map}(X, S^1).$$

Moreover, $\hat{A} = d_{\mathbb{R}} \oplus A_L$ on $\mathfrak{g}_{\mathfrak{t}} \cong \underline{\mathbb{R}} \oplus L$.

<u>STRATIFICATION</u>:

IRREDUCIBLE CONN'S
NON-ZERO SPINORS

$$\mathcal{M}_t \cong M^w_\kappa \sqcup \mathcal{M}^{*,0}_t \sqcup \bigsqcup_{\{s : s \oplus s' = t\}} M_s$$

OPEN, SMOOTH $\subset \mathcal{M}_t$

Donaldson invariants

Set $\mathbf{A}(X) = \mathrm{Sym}(H_0(X;\mathbf{R}) \oplus H_2(X;\mathbf{R}))$, so $z \in \mathbf{A}(X)$ is a linear sum of **monomials**

$$x^m \beta_1 \beta_2 \cdots \beta_{\delta-2m},$$

with $x \in H_0(X;\mathbf{Z})$ positive generator and $\beta_i \in H_2(X;\mathbf{R})$. Useful cohomology classes on M^w_κ can be defined via

$$\mu : H_i(X;\mathbb{R}) \to H^{4-i}(M^w_\kappa;\mathbb{R}).$$

The Donaldson invariant is a **linear function**

$$D^w_X : \mathbf{A}(X) \to \mathbf{R},$$

where, for a monomial z with $\deg(z) = 2\delta$,

$$D^w_X(z) = \langle \mu(z), [M^w_\kappa] \rangle$$

with $\mu(z) = \mu(x)^m \smile \mu(\beta_1) \smile \cdots \smile \mu(\beta_{\delta-2m})$.

X has *KM-simple type* if for some w, and all $z \in \mathbf{A}(X)$,

$$D^w_X(x^2 z) = 4D^w_X(z),$$

One defines the *Donaldson series*:

$$\mathbf{D}^w_X(h) = D^w_X((1+\tfrac{1}{2}x)e^h), \quad h \in H_2(X;\mathbb{R}).$$

Theorem 1 (Kronheimer & Mrowka, 1993). *If X has KM-simple type, then there exist $a_r \in \mathbb{Q}$ and $K_r \in H^2(X;\mathbb{Z})$ (the KM-basic classes), so*

$$\mathbf{D}^w_X(h) = e^{\frac{1}{2}Q_X(h,h)} \sum_{r=1}^{s} (-1)^{\frac{1}{2}(w^2+w\cdot K_r)} a_r e^{K_r}.$$

Seiberg-Witten invariants and Witten's conjecture

$M_{\mathfrak{s}}$ is compact manifold with, for $c_1(\mathfrak{s}) = c_1(W^+)$,

$$\dim M_{\mathfrak{s}} = d_{\mathfrak{s}}(\mathfrak{s}) = \tfrac{1}{4}(c_1(\mathfrak{s})^2 - 2\chi - 3\sigma).$$

Seiberg-Witten invariants: $SW_X : \mathrm{Spin}^c(X) \to \mathbb{Z}$,

$$SW_X(\mathfrak{s}) = \langle \mu(x)^{d_{\mathfrak{s}}/2}, [M_{\mathfrak{s}}] \rangle$$

where $\mu : H_0(X;\mathbb{Z}) \to H^2(M_{\mathfrak{s}};\mathbf{Z})$ defines classes on $M_{\mathfrak{s}}$.

X has *SW-simple type* if

$c_1(W^+)$

$$SW_X(\mathfrak{s}) = 0 \quad \text{when} \quad \dim M_{\mathfrak{s}}(X) > 0.$$

Call $c_1(\mathfrak{s}) \in H^2(X)$ an *SW-basic class* if $SW_X(\mathfrak{s}) \neq 0$.

Seiberg-Witten series: For $h \in H_2(X;\mathbb{R})$, define

$$\mathbf{SW}_X^w(h) = \sum_{\mathfrak{s}\in\mathrm{Spin}^c(X)} (-1)^{\frac{1}{2}(w^2 + c_1(\mathfrak{s})\cdot w)} SW_X(\mathfrak{s}) e^{\langle c_1(\mathfrak{s}), h\rangle}.$$

This is a finite sum. Let $c(X) = -\frac{1}{4}(7\chi + 11\sigma)$, where X has Euler characteristic χ and signature σ.

Conjecture 2 (Witten). A 4-manifold X has KM-simple type if and only if it has SW-simple type; if X has simple type, the KM and SW basic classes coincide, and

$$\mathbf{D}_X^w(h) = 2^{2-c(X)} e^{\frac{1}{2}Q_X(h,h)} \mathbf{SW}_X^w(h), \quad h \in H_2(X;\mathbb{R}).$$

Bubbling and Uhlenbeck compactness

If $\{A_\alpha, \Phi_\alpha\}_{\alpha \in \mathbb{N}} \subset \tilde{\mathcal{M}}_{\mathfrak{t}}$, then it converges to *ideal PU(2) monopole* $(A_\infty, \Phi_\infty, \mathbf{x})$ in $\tilde{M}_{\mathfrak{t}_\ell} \times \mathrm{Sym}^\ell(X)$ if

- $(A_\alpha, \Phi_\alpha) \to (A_\infty, \Phi_\infty)$ in C^∞ on $X \setminus \{\mathbf{x}\}$, mod gauge,
- $|F_{A_\alpha}|^2 \to |F_{A_\infty}|^2 + 8\pi^2 \sum_{x \in \mathbf{x}} \delta(x)$, as measures.

Let $\bar{\mathcal{M}}_{\mathfrak{t}}$ be closure of $\mathcal{M}_{\mathfrak{t}}$ w.r.t. *Uhlenbeck topology* in

$$\bigsqcup_{\ell=0}^{N} \left(M_{\mathfrak{t}_\ell} \times \mathrm{Sym}^\ell(X)\right),$$

where $\mathfrak{t}_\ell = (\rho, V_\ell)$ with

$$p_1(\mathfrak{t}_\ell) = p_1(\mathfrak{t}) + 4\ell,$$

while $w_2(\mathfrak{t}_\ell) = w_2(\mathfrak{t})$ and $c_1(\mathfrak{t}_\ell) = c_1(\mathfrak{t})$.

Theorem 4 (F-Leness, 1996). *$\bar{\mathcal{M}}_{\mathfrak{t}}$ is compact.*

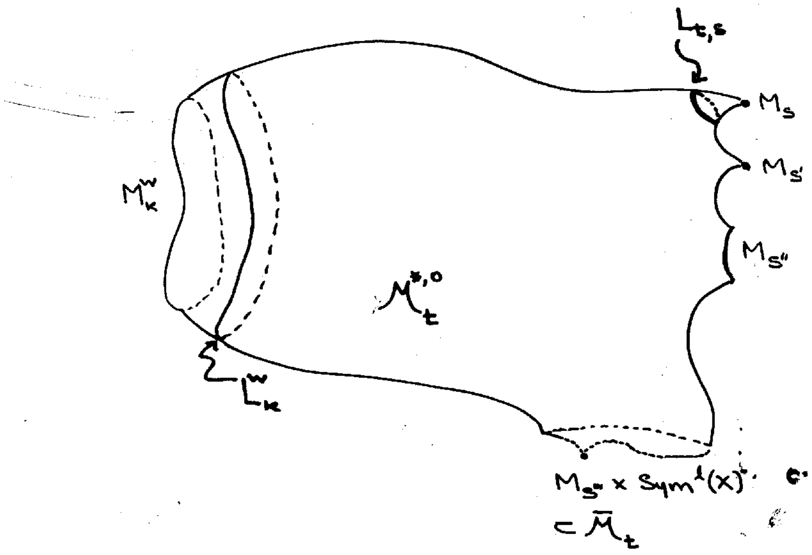
The space $\bar{\mathcal{M}}_{\mathfrak{t}}$ is smoothly stratified, with

- *top* or *zeroth level* $\mathcal{M}_{\mathfrak{t}}$, and
- *lower levels* $M_{\mathfrak{t}_\ell} \times \mathrm{Sym}^\ell(X)$, $\ell \geq 1$.

Need analytical tools ("gluing theory") to understand

- Topology of "tubular neighborhoods" of the boundary strata and singularities, and
- How strata $M_{\mathfrak{t}_\ell} \times \Sigma$ fit together, for all $\Sigma \subset \mathrm{Sym}^\ell(X)$.

PU(2)-MONOPOLE COBORDISM



$$\begin{tikzcd}
\mathbb{F}_t = \tilde{\mathcal{C}}_t^{*,0}/S^1 \times_{\mathcal{G}_t} \mathfrak{g}_t \arrow[d] \\
\mathcal{C}_t^{*,0}/S^1 \times X
\end{tikzcd}$$

$$\mu_{p_1}(\beta) = -\tfrac{1}{4}p_1(\mathbb{F}_t)/\beta \in H^{4-\bullet}(\mathcal{C}_t^{*,0}/S^1)$$

$$\beta \in H_\bullet(X)$$

$$\begin{tikzcd}
\mathbb{L}_t = \mathcal{C}_t^{*,0} \times_{(S^1, \times -2)} \mathbb{C} \arrow[d] \\
\mathcal{C}_t^{*,0}/S^1
\end{tikzcd}$$

$$\mu_{c_1} = c_1(\mathbb{L}_t)$$

PU(2)-monopole cobordism formula relating D and SW invariants

– Modulo the (non-trivial) construction of links $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$, the cobordism $\mathcal{M}_{\mathfrak{t}}^{*,0}$ yields the raw identity,

$$\begin{aligned}&\langle \mu_{p_1}(z) \smile \mu_{c_1}^{n_a-1}, [\mathbf{L}^{w}_{\mathfrak{t},\kappa}]\rangle \\ &= - \sum_{\mathfrak{s}\in \mathrm{Spin}^c(X)} \langle \mu_{p_1}(z) \smile \mu_{c_1}^{n_a-1}, [\mathbf{L}_{\mathfrak{t},\mathfrak{s}}]\rangle ,\end{aligned}$$

where

$$\deg(z) = d_a = \dim M_{\kappa}^{w} \text{ and } d_a + 2n_a = \dim \mathcal{M}_{\mathfrak{t}}.$$

- One finds that $\langle \mu_{p_1}(z) \smile \mu_{c_1}^{n_a-1}, [\mathbf{L}^{w}_{\mathfrak{t},\kappa}]\rangle$ is a multiple of Donaldson invariant, $D_X^w(z)$.
- Difficult part is to show $\langle \mu_{p_1}(z) \smile \mu_{c_1}^{n_a-1}, [\mathbf{L}_{\mathfrak{t},\mathfrak{s}}]\rangle$ is correct multiple of Seiberg-Witten invariant $SW_X(\mathfrak{s})$.

APPLICATION OF PU(2) MONOPOLE COBORDISM IN SIMPLEST SITUATIONS

(-1) NO SW MODULI SPACES WITH NON-ZERO INVARIANTS APPEAR IN $\bar{\mathcal{M}}_t$:

$$D^w_X(h) \equiv 0 \pmod{h^{c(X)-2}}$$

(0) SW MODULI WITH NON-ZERO INVARIANTS APPEAR IN $\mathcal{M}_t$ (TOP LEVEL OR LEVEL 0) ONLY:

WITTEN'S FORMULA HOLDS $(\mathrm{mod}\ h^{c(X)})$.

* (1) $M_s \times X \subset \bar{\mathcal{M}}_t$, $SW_X(s) \neq 0$

WITTEN'S FORMULA $(\mathrm{mod}\ h^{c(X)+2})$

(ℓ) $M_s \times \mathrm{Sym}^\ell(X) \subset \bar{\mathcal{M}}_t$, $SW_X(s) \neq 0$

WITTEN'S FORMULA $(\mathrm{mod}\ h^{c(X)+2\ell})$

APPLICATION OF COBORDISM IN SIMPLEST CASES ($M_s \times Sym^\ell(X) \subset \bar{\mathcal{M}}_t$, $\ell = 0,1,2$) IMPOSES CONSTRAINTS ON 4-MANIFOLDS, SO THAT COBORDISM YIELDS USEFUL RESULTS:

$\mathcal{B}$ = {SW-BASIC CLASSES} $\subset H^2(X, \mathbb{Z})$

$\mathcal{B}^\perp$ = Q_X-ORTHOGONAL COMPLEMENT OF $\mathcal{B}$

A FOUR-MANIFOLD X IS ABUNDANT IF $Q_X|_{\mathcal{B}^\perp}$ CONTAINS HYBERBOLIC SUBLATTICE, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

THEOREM EVERY COMPACT COMPLEX ALGEBRAIC SIMPLY CONNECTED SURFACE, $b_2^+ \geq 3$, ABUNDANT.

"LEVEL-ZERO" CASE

SW MODULI SPACES WITH NON-ZERO INVARIANTS ONLY APPEAR IN TOP LEVEL, M_t, OF $\bar{M}_t \subset \coprod_{\ell \geq 0} (M_{t_\ell} \times \mathrm{Sym}^\ell(X))$.

THEOREM ASSUME X HAS SW-SIMPLE TYPE, ABUNDANT, $b_1 = 0$, ODD $b_2^+ > 1$, $\Lambda \in B^\perp$ EXISTS WITH $\Lambda^2 = 2-(\chi_h+\sigma)$, $w \in H^2(X,\mathbb{Z})$ WITH $w - \Lambda \equiv w_2(X)$ (mod 2). FOR ALL $h \in H_2(X;\mathbb{R})$,

$$\boxed{\begin{aligned} {}^{*}\mathcal{D}^w_X(h) &\equiv 0 \equiv SW^w_X(h) \pmod{h^{c(X)-2}} \\ \mathcal{D}^w_X(h) &\equiv 2^{2-c(X)} e^{\frac{1}{2}Q_X(h,h)} SW^w_X(h) \\ &\qquad \pmod{h^{c(X)}} \end{aligned}}$$

* JOINT W/ KRONHEIMER-MROWKA

"LEVEL-ONE" CASE

ALLOW

$$M_s \times \mathrm{Sym}^{\ell}(X) \subset \overline{\mathcal{M}}_{t'}$$

WITH $SW_X(s) \neq 0$ ONLY IF

$\ell = 0$ OR $\ell = 1$.

<u>THEOREM</u> SAME HYPOTHESES AS BEFORE, EXCEPT ASSUME $\Lambda \in B^{\perp}$ HAS $\Lambda^2 = 4 - (\chi + \sigma)$. FOR ALL $h \in H_2(X;\mathbb{R})$,

$$\boxed{D^{w}_{X}(h) \equiv 2^{2-c(X)} e^{\frac{1}{2} Q_X(h,h)} SW_X(h) \pmod{h^{c(X)+2}}}$$

REMARK: LAST CASE CAN DO BY DIRECT CALCULATION, "$\ell = 2$", GIVES WITTEN'S FORMULA MOD $h^{c(X)+4}$, WITH $\Lambda^2 = 6 - (\chi + \sigma)$.

A strategy to prove Witten's conjecture

1. PU(2)-monopole cobordism, with gluing construction of links $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ of $M_{\mathfrak{s}} \times \mathrm{Sym}^{\ell}(X)$ in $\bar{\mathcal{M}}_{\mathfrak{t}}$ yields
$$D_X^w(z) = 2^{2-c(X)} \sum_{\mathfrak{s}\in \mathrm{Spin}^c(X)} (-1)^{\frac{1}{2}(w^2+c_1(\mathfrak{s})\cdot w)} \times \langle \mu_{p_1}(z) \smile \mu_{c_1}^{n_a-1}, [\mathbf{L}_{\mathfrak{t},\mathfrak{s}}]\rangle.$$

2. Pairings with $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ ***hard to compute directly*** when $\ell = \ell(\mathfrak{t},\mathfrak{s}) > 2$. Instead try to show:
   - Pairings
   $$\langle \mu_{p_1}(z) \smile \mu_{c_1}^{n_a-1}, [\mathbf{L}_{\mathfrak{t},\mathfrak{s}}]\rangle$$
   are universal polynomials in Q_X, $\langle c_1(\mathfrak{s}), \cdot\rangle$ and $\langle \Lambda, \cdot\rangle$, with coeffs depending on $SW_X(\mathfrak{s})$, $c_1(\mathfrak{s})$, Λ, χ, σ.
   - Above is already enough to prove
   $$\text{Seiberg-Witten} \implies \text{Donaldson.}$$

3. We expect our universal, but non-explicit formula implies *Witten's formula* via supplementary arguments,
   - Known Donaldson and Seiberg invariants of sufficiently many examples,
   - Recursion relations among coefficients.

Formulas for D invariants due to SW moduli spaces in arbitrary lower levels

More generally, our construction of links and a push-forward/integration-over-fibers argument yields

Conjecture 11 (F-Leness, March 1999). For Λ, w in $H^2($
with $w - \Lambda \equiv w_2(X) \pmod 2$, and all $h \in H^2(X;\mathbb{R})$,

$$D^w_X(h^d) = 2^{2-c(X)} \sum_{\mathfrak{s} \in \mathrm{Spin}^c(X)} (-1)^{\frac{1}{2}(w^2 + c_1(\mathfrak{s})\cdot w)} SW_X(\mathfrak{s})$$

$$\times \sum_{i=0}^{m} \boldsymbol{\delta}_i \left(\langle c_1(\mathfrak{s}), h\rangle, \langle \Lambda, h\rangle\right) Q_X^{\ell - i}(h,h),$$

where $\ell = \ell(c_1(\mathfrak{s}), \Lambda)$ and $m(\ell, d) = \min(\ell, [d/2])$ above. The functions $\boldsymbol{\delta}_i$ are homogeneous polynomials in two variables, whose coefficients are polynomials in $2\chi \pm 3\sigma$, $(c_1(\mathfrak{s}) - \Lambda)^2$, Λ^2, and $(c_1(\mathfrak{s} - \Lambda) \cdot c_1(\mathfrak{s})$. A similar formula holds for the invariants $D^w_X(x^m h^{d-2m})$, $0 \le m \le [d/2]$.

Remark 12. One sees a factorization of the pairings

$$\langle \mu_{p_1}(z) \cup \mu_{c_1}^{\delta_{c_1}}, [\mathbf{L}_{\mathfrak{t},\mathfrak{s}}]\rangle$$

into product of $SW_X(\mathfrak{s})$ and $\boldsymbol{\delta} = \sum_{i=1}^{m} \boldsymbol{\delta}_i Q_X^{\ell - i}$.

For $M_{\mathfrak{s}}$ a single point and $M_{\mathfrak{s}} \hookrightarrow \mathcal{M}_{\mathfrak{t}_\ell}$ a smooth submanifold (no obstructions), the SW link pairings have (essentially) same structure as those in wall-crossing terms for Donaldson invariants when $b_2^+(X) = 1$.

IDEAS UNDERLYING PROOF OF WITTEN'S FORMULA USING COBORDISM:

WE'LL DESCRIBE HOW TO COMPUTE PAIRINGS

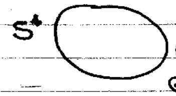
$$\langle \mu_p(z) \cup \mu_c^{n_a - 1}, [\mathbb{L}_{t,s}] \rangle.$$

THE SW CONTRIBUTIONS, WHEN LEVEL, $\boxed{\ell = 1}$, AND ONE SHOULD BE ABLE TO GENERALIZE TO ARBITRARY $\boxed{\ell \geqslant 0}$.

TOPOLOGICAL MODELS FOR LINKS OF SEIBERG-WITTEN STRATA

- USE GLUING THEORY TO CONSTRUCT TOPOLOGICAL MODEL FOR NEIBHBORHOOD OF, SEIBERG-WITTEN STRATUM
  $$M_s \times \mathrm{Sym}^\ell(X) \subset \bar{\mathcal{M}}_t$$
- GIVEN TOPOLOGICAL MODEL FOR LINK $\mathbb{L}_{t,s}$ OF $M_s \times \mathrm{Sym}^\ell(X)$, CAN THEN TRY TO APPLY STANDARD METHODS OF INTERSECTION THEORY .

ONE-BUBBLE GLUING

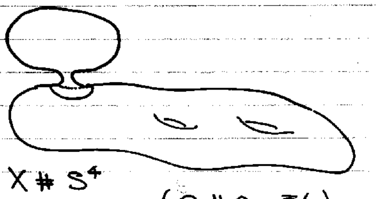


$(A_1, 0)$

$\mathcal{G}(A_1, 0) = 0$

(A_0, Φ_0)

$\mathcal{G}(A_0, \Phi_0) = 0$

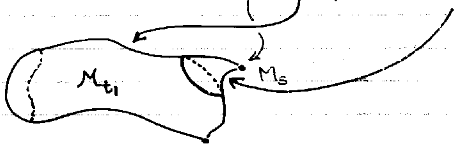


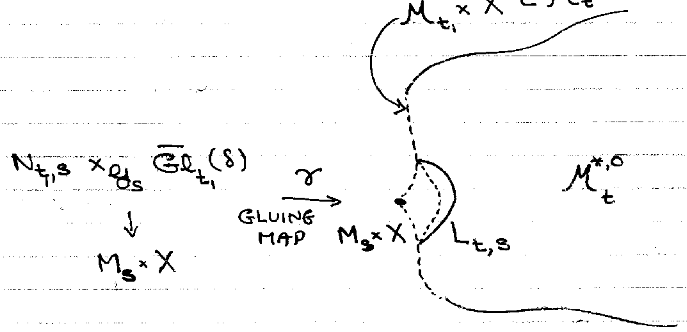
$(A_0 \# A_1, \Phi_0')$

$\mathcal{G}(A_0 \# A_1, \Phi_0') \approx 0$

$p_1(t_1) = p_1(t) + 4$

$[A_0, \Phi_0] \in N(M_s, \mathcal{M}_{t_1}) \cong T(M_s, \mathcal{M}_{t_1}) \subset \mathcal{M}_{t_1}^{C}$





GLUING MODEL FOR NEIGHBORHOOD OF $M_s \times Sym^\ell(X)$

$$\boxed{\mathcal{M}^{vir}_{t,s} = \widetilde{N}^{(\epsilon)}_{t_\ell,s} \times_{\mathcal{G}_s} Gl_{t_\ell}(\delta)}$$

$N_{t_\ell,s} \longrightarrow M_s$ "VIRTUAL" NORMAL BUNDLE OF $M_s \subset \mathcal{M}_{t_\ell}$

$\Xi_{t_\ell,s} \longrightarrow M_s$ "OBSTRUCTION" BUNDLE FOR $M_s \subset \mathcal{M}_{t_\ell}$.

$$[N_{t_\ell,s}] - [\Xi_{t_\ell,s}] \in K(M_s)$$

$$M_s \times Sym^\ell(X) \subset \mathcal{M}_{t_\ell} \times Sym^\ell(X) \subset \overline{\mathcal{M}}_t$$

$N_{t_\ell,s}$ WOULD BE TRUE NORMAL BUNDLE IF $M_s \subset \mathcal{M}_{t_\ell}$ SUBMANIFOLD.

UHLENBECK COMPACTIFICATION OF GLUING MODEL:

$M_1^{s,*}(S^4,\delta) \cong (0,\delta] \times SO(3)$

$\bar{M}_1^{s,*}(S^4,\delta) = M_1^{s,*} \cup \{N\} \times [\Theta]$

$\cong c(SO(3))$ (CONE)

$\overline{Gl}_{t_1}(\delta) = Gl_{t_1}(\delta)$ WITH $\bar{M}_1^{s,*}(S^4,\delta)$ IN PLACE OF $M_1^{s,*}(S^4,\delta)$

$$\bar{\mathcal{M}}^{vir}_{t,s} = \tilde{N}_{t_\ell,s} \times_{\mathcal{G}_s} \overline{Gl}_{t_\ell}(\delta)$$

$$\overline{Gl}_{t_1}(\delta) = Gl_{t_1}(\delta) \sqcup X$$

MODEL STRATIFICATION:

$$\bar{\mathcal{M}}^{vir}_{t,s} = \mathcal{M}^{vir}_{t,s} \sqcup (N_{t,s}(\varepsilon) - M_s) \times X \sqcup M_s \times X$$

NEXT STEP IS CONSTRUCTION OF LINK, $\mathbb{L}_{t,s}$, OF $M_s \times X$.

($M_s \times \mathrm{Sym}^\ell(X)$, IN GENERAL)

LINK OF SEIBERG-WITTEN STRATUM

SW COMPONENT OF MODEL, "VIRTUAL" LINK:

$$\mathbb{L}^{s,vir}_{t,\mathfrak{s}} = \left(\partial \tilde{N}_{t_\ell,\mathfrak{s}}(\varepsilon) \times_{\ell\gamma_s} \overline{\mathrm{Gl}}_{t_\ell}(\delta)\right)/S^1$$

INSTANTON COMPONENT OF MODEL, "VIRTUAL" LINK:

$$\mathbb{L}^{i,vir}_{t,\mathfrak{s}} = \left(\tilde{N}_{t_\ell,\mathfrak{s}}(\varepsilon) \times_{\ell\gamma_s} \partial \overline{\mathrm{Gl}}_{t_\ell}(\delta)\right)/S^1$$

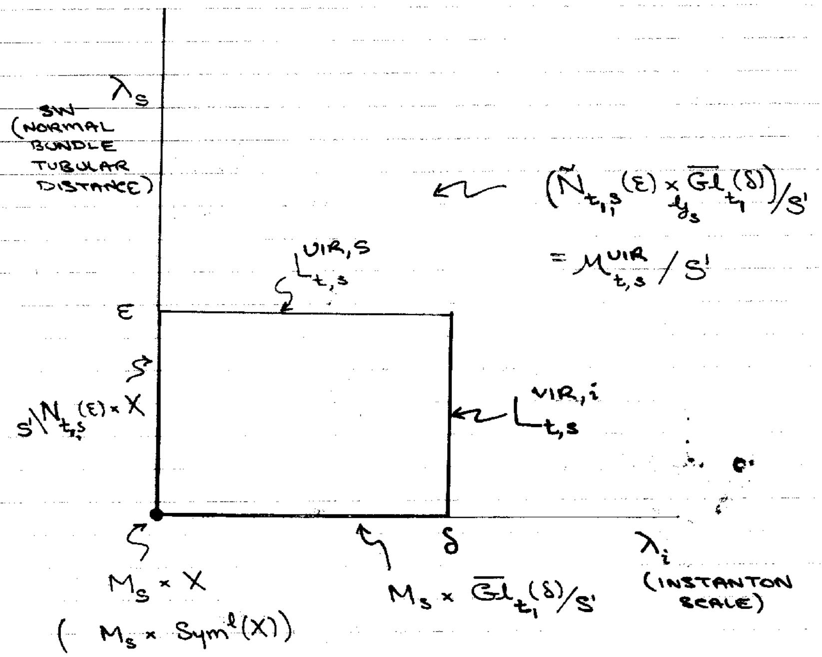
"VIRTUAL" LINK OR LINK OF $M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X) \subset \bar{\mathcal{M}}^{vir}_{t,\mathfrak{s}}/S^1$:

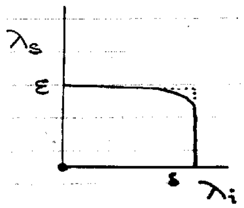
$$\mathbb{L}^{vir}_{t,\mathfrak{s}} = \mathbb{L}^{vir,i}_{t,\mathfrak{s}} \cup \mathbb{L}^{vir,s}_{t,\mathfrak{s}}$$

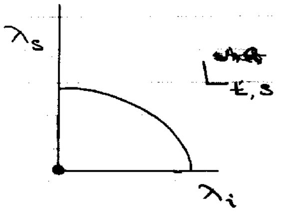
LINK OF $M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X) \subset \bar{\mathcal{M}}_t/S^1$:

$$\begin{aligned}\mathbb{L}_{t,\mathfrak{s}} &= \gamma\left(\chi^{-1}(0) \cap \mathbb{L}^{vir}_{t,\mathfrak{s}}\right)\\ &= \left(\bar{\mathcal{M}}_t/S^1\right) \cap \gamma\left(\mathbb{L}_{t,\mathfrak{s}}\right)\end{aligned}$$

Model SW Link







"INSTANTON COMPONENT" OF GLUING MODEL:

$\boxed{\ell = 1}$

$$Gl_{t_\ell}(\delta) = \frac{\left(Fr(g_{t_\ell}) \times_X Fr(T^*X) \times M_1^{s,\#}(S^4,\delta)\right)}{SO(3) \times SO(4)}$$

$M_k^s(S^4)$: MODULI SPACE OF k-INSTANTONS ON S^4, FRAMED AT S-POLE

$M_k^{s,\#}(S^4,\delta)$: $\#$ ~ CENTER OF MASS AT N-POLE.

δ ~ SCALE $\leq \delta$

$\boxed{\ell \geq 1}$

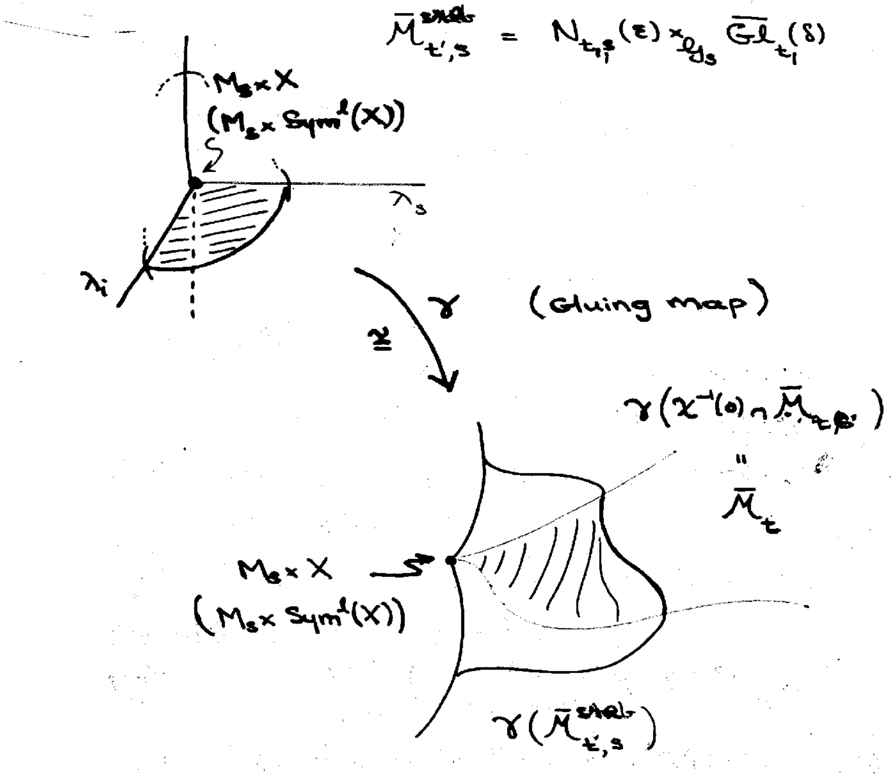
$Gl_{t_\ell}(\delta, \Sigma) \longrightarrow \Sigma$, BUNDLE WITH INSTANTON MODULI FIBERS

$\Sigma \subset Sym^\ell(X)$ A SMOOTH STRATUM.

$$Gl_{t_\ell}(\delta) \sim \text{"} \bigcup_{\Sigma \subset Sym^\ell(X)} Gl_{t_\ell}(\delta,\Sigma) \to Sym^\ell(X) \text{"}.$$

Effect of the gluing map

$$\bar{\mathcal{M}}^{\text{stab}}_{t',s} = N_{t_1,s}(\varepsilon) \times_{\mathcal{G}_s} \overline{Gl}_{t_1}(\delta)$$



GLUiNG FOR LEVEL ℓ

$$\mathrm{Sym}^\ell(X) \quad \begin{matrix}\Sigma_0\\ \Sigma_1\\ \vdots\end{matrix}$$

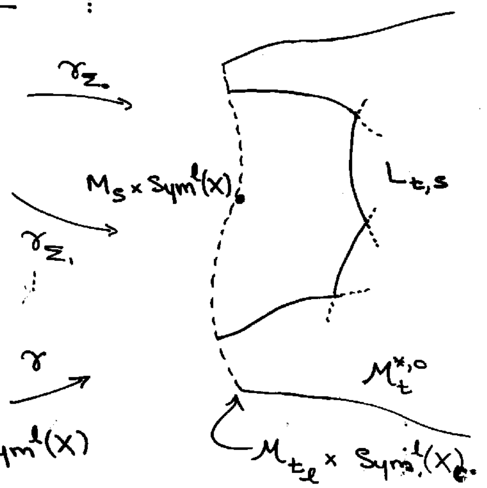
$N(M_s, \mathcal{M}_{t_\ell}) \times Gl_{\Sigma_0}$

$N(M_s, \mathcal{M}_{t_\ell}) \times Gl_{\Sigma_1}$

$\vdots$

$N(M_s, \mathcal{M}_{t_\ell}) \times Gl_{\mathrm{Sym}^\ell(X)}$

(ASSEMBLED FROM DIFFERENT γ_Σ)



$\subset \bar{\mathcal{M}}_t$

$p_1(t_\ell) = p_1(t) + 4\ell$

EG., $\mathrm{sym}^2(X) = \Sigma_0 \cup \Sigma_1$, $\Sigma_0 = \{2 \text{ distinct pts}\}$

$\Sigma_1 = \{1 \text{ pt mult } 2\}$

$= \mathrm{DIAG}(X \times X)$

CONTRIBUTION OF SW CALCULATION:

$$\left\langle \gamma^* \mu_p(z) \cup \gamma^* \mu_c^{\delta_c} \cup e_s \cup e_i , [\mathbb{L}^{\mathrm{VIR}}_{t,s}] \right\rangle$$

e_s, e_i : EULER CLASSES OF SW, INSTANTON COMPONENTS OF OBSTRUCTION BUNDLE OVER $\mathcal{M}^{\mathrm{VIR}}_{t,s}$.

ARGUMENT INVOLVING THOM CLASSES EXPRESSES PAIRING WITH $[\mathbb{L}^{\mathrm{VIR}}_{t,s}]$ AS PAIRING WITH $[M_s \times \partial\overline{Gl}_{t_\ell}(\delta)/S^1]$

PAIRING WITH $[M_s \times \partial\overline{Gl}_{t_\ell}(\delta)/S^1]$

$$\Longleftrightarrow \begin{cases} \text{PAIRINGS WITH } [M_s] \Leftrightarrow SW_X(s) \\ \times \\ {}^*\text{PAIRINGS WITH } [\partial\overline{Gl}_{t_\ell}(\delta)/S^1] . \end{cases}$$

* DIFFICULT BY DIRECT CALCULATION WHEN $\ell \geq 3$.

REMARKS

A KEY REMAINING POINT IS TO COMPUTE PAIRINGS WITH $[\partial \bar{G}\ell_{t\ell}(\delta)]$ FOR $\ell \geq 1$.

- EXPLOITING KNOWN RESULTS FOR D, SW SERIES OF ALGEBRAIC SURFACES

FOR EXAMPLE, WHEN $\ell = 1$, ONLY 3 PAIRINGS NEEDED. INDEPENDENT OF WHETHER X "SIMPLE TYPE" OR NOT: UNIVERSAL.

- ANALOGUES OF GÖTTSCHE-KOTSCHICK-MORGAN WALL-CROSSING FORMULAS FOR D INVARIANTS WHEN $b_2^+ = 1$.

HIGHER-DIM FAMILIES (OF METRICS,...) WHEN $b_2^+ > 1$?

- LEFSCHETZ FIBRATIONS?

$$\begin{tikzcd} \Sigma \arrow[r] & X^4 \arrow[d] \\ & S \end{tikzcd}$$

Σ ~ GENERIC FIBER

TOY MODEL: $X = \Sigma \times S$

- $SO(3)$ - INSTANTONS OVER $\Sigma \times S$

  ~ HOLOMORPHIC MAPS, $S \to M_\Sigma$,

  M_Σ = MODULI SPACE OF FLAT $SO(3)$ CONNECTIONS ON Σ

- SW $U(1)$-MONOPOLES OVER $\Sigma \times S$

  ~ HOLOMORPHIC MAPS, $S \to M_{\Sigma,d}(J,\tau)$

$M_{\Sigma,d}(J,\tau)$ = MODULI SPACE OF VORTICES ON $L \to \Sigma$,
$d = \langle c_1(L), [\Sigma] \rangle$,

PAIRS $(C, \varphi) \in \mathcal{A}_L \times C^\infty(L)$

- $SO(3)$ - MONOPOLES OVER $\Sigma \times S$

  ~ HOLOMORPHIC MAPS FROM S INTO MODULI SPACE OF NON-ABELIAN VORTICES ON Σ.

ABOVE IDENTIFICATIONS CAN BE ACHIEVED VIA

- ADIABATIC LIMIT ANALYSIS,

  $g = g_\Sigma \oplus g_S$ ON $\Sigma \times S$

  $\mapsto g_\varepsilon = \varepsilon^2 g_\Sigma \oplus g_S$, $\varepsilon \to 0$.

OR

- RESTRICTION OF STABLE, HOL'C BUNDLES OVER $\Sigma \times S$ TO $\Sigma \times \{z\}$, $z \in S$ VARIES.

MANY POINTS TO ADDRESS:

- RELATIONSHIP BETWEEN COMPACTIFICATIONS?
- EXTENSION OF PICTURE FOR $\Sigma \times S$ TO
  - NON TRIVIAL SURFACE BUNDLES
  - LEFSCHETZ FIBRATIONS?