GABAI

Theorem If M is a closed hyperbolic 3-manifold then $\mathrm{Isom}(M) \overset{h.e.}{\simeq} \mathrm{Diff}(M)$

Facts 1) $\mathrm{Diff}(M)$ has the homotopy type of a CW complex

2) $|\mathrm{Isom}(M)| < \infty$

$\therefore$ **Need to show**

1) $\mathrm{Isom}(M) \overset{1\text{-}1}{\longleftrightarrow} \pi_0(\mathrm{Diff}(M))$

2) $\pi_n(\mathrm{Diff}_0(M)) = 0$ all $n \geq 1$

S1

Theorem [G 1993], [GMT 1996]

$$\mathrm{Isom}(M) \overset{1\text{-}1}{\longleftrightarrow} \pi_0(\mathrm{Diff}(M))$$

if M closed, hyperbolic

$\pi_1(\mathrm{Diff}(M)) = 0$

Theorem (Hatcher 1976 / Ivanov)

$$M \text{ Haken} \Rightarrow \mathrm{Diff}_0(M) \simeq *$$

Hyperbolic

(Waldhausen 1968)
(Mostow 1968)

$$M \begin{array}{l}\text{Haken}\\ \text{Hyperbolic}\end{array} \Rightarrow \pi_0(\mathrm{Diff}(M)) \overset{1\text{-}1}{\longleftrightarrow} \mathrm{Isom}(M)$$

Hatcher's Theorem (1983) (Smale Conj.)

$$O(4) = \mathrm{Isom}(S^3) \simeq \mathrm{Diff}(S^3)$$

$\pi_0(\mathrm{Diff}^+(S^3)) = 0$ Cerf (1962) $\Gamma_4 = 0$

$$\Sigma^4 = B^4_{std} \cup_\partial B^4_{std} \Rightarrow \Sigma =_{diff} S^4$$

Thurston Geometrization conjecture

M^3 closed, ORIENTABLE irreducible, then either

i) $M = S^3/\Gamma \quad \Gamma \subset Isom_+(S^3)$

ii) $\mathbb{Z} \oplus \mathbb{Z} \subset \pi_1(M)$

iii) $M = \mathbb{H}^3/\Gamma \quad \Gamma \subset Isom_+(\mathbb{H}^3)$

Generalized Smale Conj.

M^3 closed, OR, irreducible constant sect. curvature

then $K=1$: $Diff(M) \simeq Isom(M)$

$K=0$: $Diff(M) \simeq Aff(M)$

$K=-1$: $Diff(M) \simeq Isom(M)$

(other cases, orbifold version)

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Coarse Torus Isotopy Theorem

M closed, orientable, hyperbolic
$\delta \subset M$ geodesic satisfying
insulator condition.

If $f: S^n \longrightarrow \mathrm{Diff}_0(M)$, then
$\exists$ cellulation Δ of B^{n+1}
and function $\underset{\Delta}{\overset{\uparrow}{\sigma}} \rightsquigarrow V_\sigma = D^2 \times S^1 \hookrightarrow M$

s.t.

i) κ proper face of σ
$\Longrightarrow V_\kappa \subset \mathring{V}_\sigma$

ii) $x \in \sigma \cap S^n \Rightarrow$
$f_x(\delta) \subset \mathring{V}_\sigma$

This result and the next imply
$\mathrm{Diff}_0(M) \simeq *$

Local Contractibility Theorem

$\delta \subset M$ a simple, oriented geod.

$V \subset M$ $V \approx D^2 \times S^1$

If $H : S^n \longrightarrow \mathrm{Diff}_0(M)$ s.t.

$H_x(\delta) \subset \mathring{V}$ all $x \in S^n$, then

H extends to a map

$G : B^{n+1} \longrightarrow \mathrm{Diff}_0(M)$ s.t.

$G_x(\delta) \subset \mathring{V}$ all $x \in B^{n+1}$.

Idea of Proof

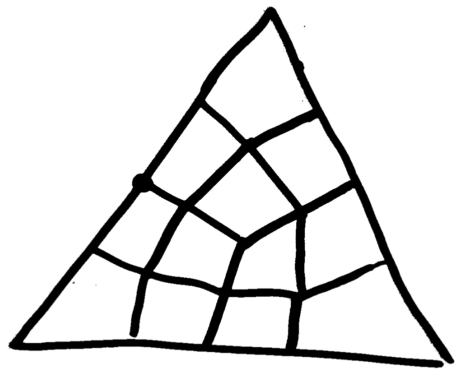
Fact: $\mathrm{Emb}_0(D^2 \times S^1, \mathbb{R}^2 \times S^1) \simeq S^1 \times S^1$

USE This to reduce to Case (cerf Palais
covering isotopy theorem
via $H_x(N(\delta)) = id_{N(\delta)}$ all $x \in S^n$

Ivanov
Now apply Hatcher (Haken Case.)

This is close to saying if $H: S^n \longrightarrow \mathrm{Emb}_0(\delta, M)$

then H is coarsely homotopically trivial



Proof of Coarse Isotopy Theorem

Canonical Solid Torus Theorem

Let δ be an oriented simple closed geodesic in M possessing a non coalescable insulator family. Associated to any Riemannian metric ρ on M $\exists$ a $V_\rho \approx D^2 \times S^1$ s.t.

1) $\mathring{V}_\rho$ is embedded $\mathring{D}^2 \times S^1$
2) each core of $\mathring{V}_\rho$ is isotopic to δ
3) If M_ρ is hyperbolic then $\delta_\rho \subset \mathring{V}_\rho$

(δ_ρ is the geod in M_ρ homotopic to δ)

If $f : S^n \to \mathrm{Diff}_0(M)$ then

f induces $h : S^n \to RM(M)$

the space of Riem. metrics on M, via $h(x) = (f_x)_* \rho_0$

where ρ_0 is a fixed hyp metric on M.

h extends to

$$H : B^{n+1} \to RM(M)$$

Since $RM(M)$ is contractible.

Thus we get function

$$\underset{\underset{B^{n+1}}{\cap}}{x} \longrightarrow \underset{\underset{V_{H(x)}}{\updownarrow \text{def}}}{\dot{V}_x} \approx \dot{D}^2 \times S^1 \hookrightarrow M$$

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These V_x do not vary continuously in x, but they satisfy

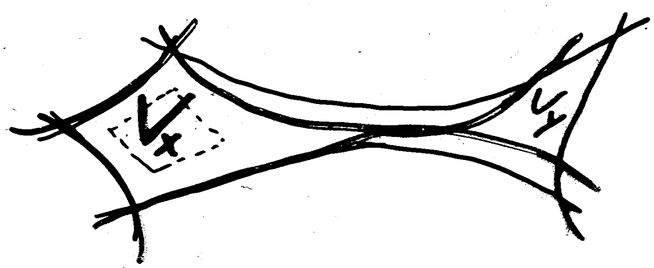
Non Encroachment ~~property~~

If $x \in B^{n+1}$ and $\epsilon > 0$

then exists $\alpha > 0$ s.t.

$$\{z \mid d_x(z, \partial V_x) \geq \epsilon \text{ and } z \in V_x\}$$

$$\subset \{z \in V_y \mid d_y(z, \partial V_y) \geq \epsilon/2\}$$

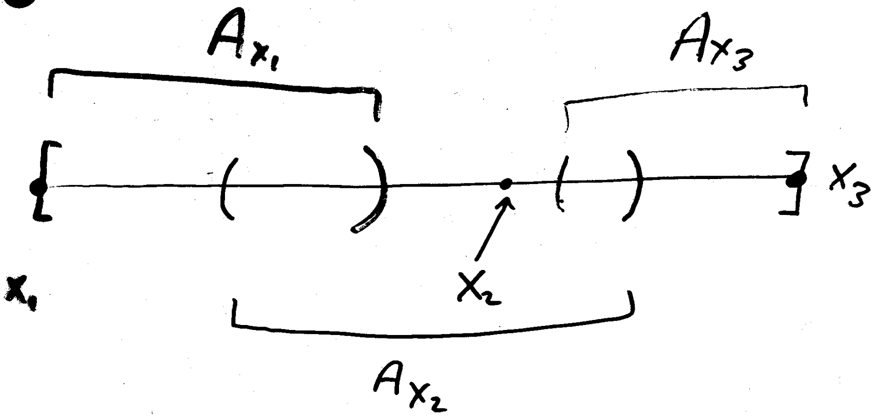


if

$d_{B^{n+1}}(x,y) < \alpha$

Non Encroachment $\Rightarrow$ Coarse Isotopy theorem

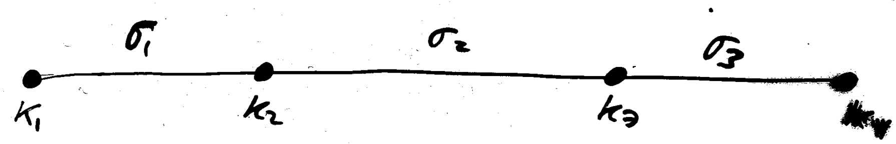
$n=0$



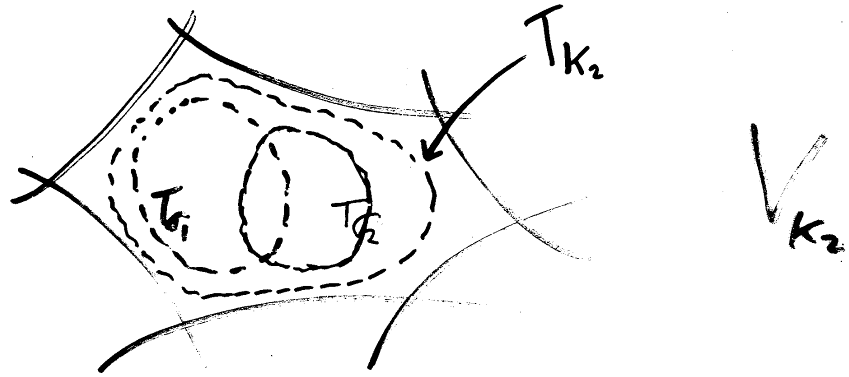
Let $T_{x_i} \approx D^2 \times S^1 \subset \mathring{V}_{x_i}$ satisfy

$$T_{x_i} \subset V_y \quad \text{all } y \in A_{x_i}$$

Δion on B^{n+1} subordinate to $\{A_{x_i}\}$



Define $T_{\sigma_1} = T_{x_1}$ $T_{\sigma_2} = T_{x_2}$, $T_{\sigma_3} = T_{x_3}$



Let $T_{\sigma_i} := T_{x_i}$ if $\sigma_i \subset A_{x_i}$

Since $T_{\sigma_1} \cup T_{\sigma_2} \subset \mathring{V}_{K_2}$

$\exists\, T_{K_2} \approx D^2 \times S^1$ $\quad T_{K_2} \subset \mathring{V}_{K_2}$ and

$$T_{\sigma_1} \cup T_{\sigma_2} \subset T_{K_2}$$

These T_λ's satisfy conclusion of Theorem except inclusion goes wrong way!!

Proof of Canonical torus Theorem

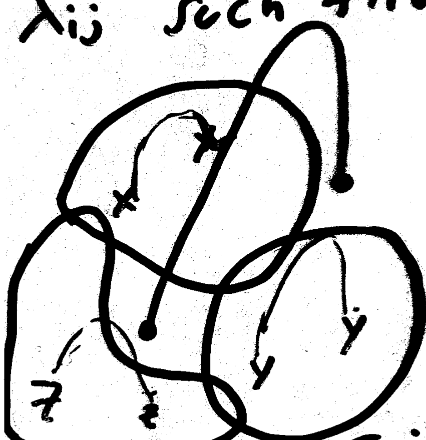
δ geodesic in M, $\quad \begin{array}{c} \mathbb{H}^3 \\ \downarrow \pi \\ M \end{array}$

$$\delta \rightsquigarrow \pi^{-1}(\delta) = \{\delta_i\} \rightsquigarrow \{\partial \delta_i\}$$

A non coalescable insulator family $\{\lambda_{ij}\}$ is a $\pi_1(M)$ invariant collection of circles λ_{ij} such that i) λ_{ij} separates $\partial\delta_i, \partial\delta_j$

ii) No trilinking

$\vdots$



Trilinking

Fix $\partial \delta_0$.

Span each λ_{0j} by a

e-LA-Lamination $\underset{\alpha}{\sigma}_{0j}$

take $H_{\underset{\alpha}{\sigma}(0,j)}$ = open comp. region containg $\partial \delta_0$

$$H_0 \stackrel{def}{=} \bigcap_{\underset{\alpha}{\sigma}(0,j)} H_{\underset{\alpha}{\sigma}(0,j)}$$

use all possible $\underset{\alpha}{\sigma}(0,j)$'s

Fact: H_0 contains a $\mathring{D}^2 \times \mathbb{R}$ component which projects to a $\mathring{D}^2 \times S^1 \stackrel{def}{=} \mathring{V}_e$

<u>Theorem</u> (G.–Meyerhoff–N. Thurston)
M closed, orientable, hyperbolic $\Rightarrow$
$\exists$ geod. satisfing the insulator cond.

Corollary The space of Hyperbolic Metrics on a hyperbolic manifold is Contractible.

Proof Let ρ_0 fixed hyp Metric

then $\Psi : \mathrm{Diff}_0(M) \xrightarrow{\cong} \mathrm{Hyp}(M)$

$\Psi(f) = f_*(\rho_0)$

Ψ 1-1 easy

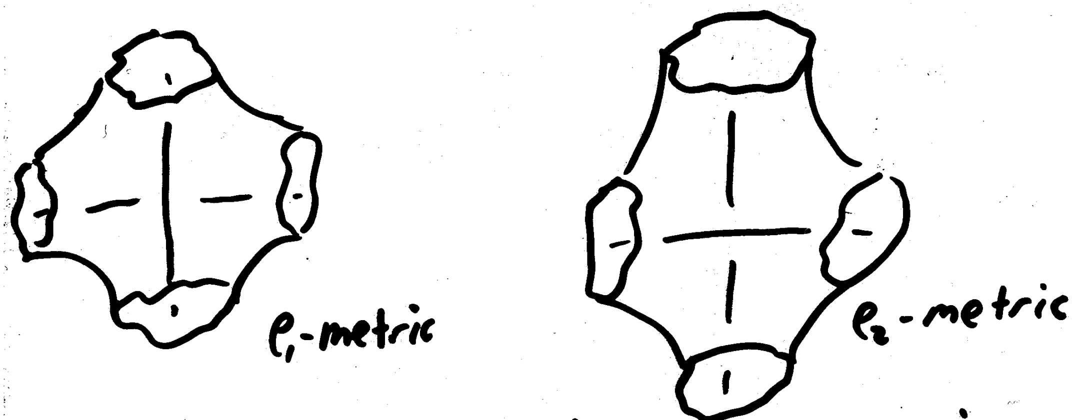
Ψ onto Mostow, [GMT] [F] $\square$

Corollary Let $\gamma \subset M$ be a s.c.c. with atoroidal Complement then
$$\mathrm{Emb}_0(\gamma, M) \simeq S^1$$

What Mostow does not Say

He says "N has a unique hyperbolic str, up to isometry homotopic to id_N."

This does not rule out



a homotopy class giving rise to distinct isotopy classes with different hyperbolic metrics.