Promoting Essential Laminations

Danny Calegari, Harvard University

Summer 2001

A codimension one foliation $\mathscr{F}$ of a 3-manifold M is *taut* if there is a transverse circle which intersects every leaf.

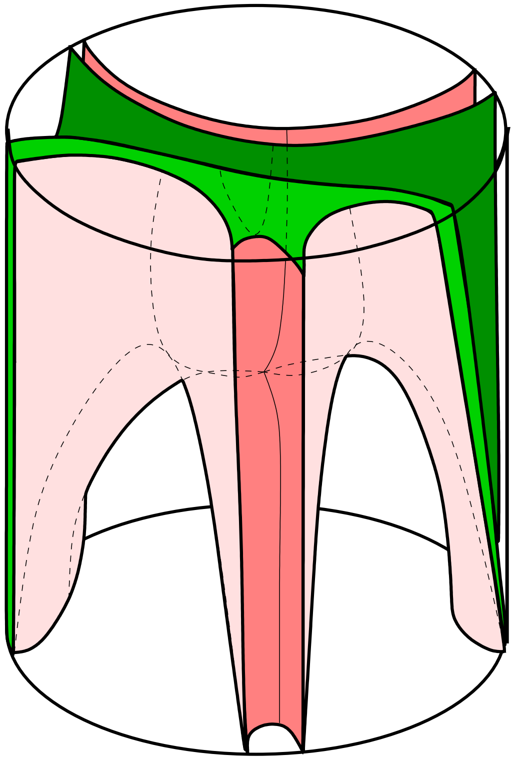
Convention: all foliations are orientable and co–orientable.

Suppose a 3–manifold M admits a taut foliation $\mathscr{F}$. Then:

- The universal cover of M is homeomorphic to $\mathbb{R}^3$ or to $S^2 \times \mathbb{R}$. [Palmeira]
- Leaves of $\mathscr{F}$ are incompressible and transverse loops to $\mathscr{F}$ are homotopically essential. [Novikov]
- There is a metric on M for which leaves of $\mathscr{F}$ are minimal surfaces. [Sullivan]

There are two views of the universal cover of a taut foliation:

1. An octopus in $\mathbb{R}^3$, where the leaves are horizontal disks
2. A solid cylinder, where the leaves bend up or down to avoid some vertical lines, and project to subdisks bounded by leaves of a pair of laminations



A *lamination* (in a manifold) is a foliation of a closed subset. It is *essential* if it has no sphere leaf or torus leaf bounding a solid torus, and if complementary regions are irreducible, and their boundary is incompressible and end–incompressible.

A complementary region decomposes into two pieces, unique up to isotopy: compact pieces called *guts*, and regions of the kind $\Sigma \times I$ called *interstitial regions*, where Σ is a noncompact surface with boundary components C_i. The annuli $C_i \times I$ are called *interstitial annuli*, and they demarcate the boundary between the guts and the interstitial regions.

A lamination is *genuine* if it is essential, and has nonempty gut.

Product regions can be blown down (collapsed). Hence an essential lamination is either genuine, or can be blown down to a taut foliation.

Manifolds with genuine laminations are very useful; if M admits a genuine lamination Λ, there are the following corollaries:

- $\pi_1(M)$ is toroidal or word–hyperbolic [Gabai–Kazez]
- The mapping class group is finite if $\pi_1(M)$ is atoroidal [Gabai–Kazez]
- many other properties . .

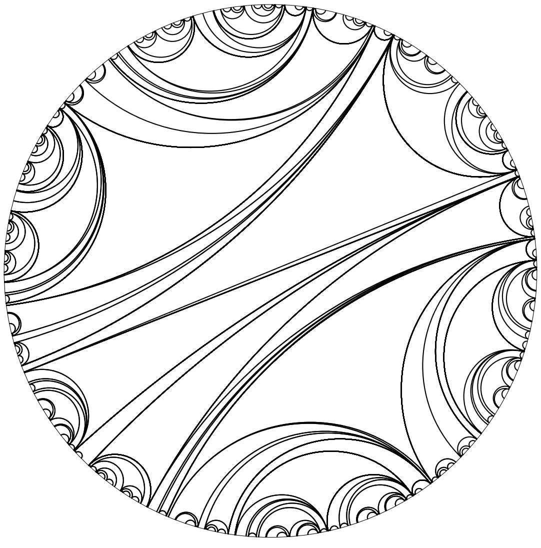
Theorem:[C-] *Let $\mathscr{F}$ be an essential lamination of an atoroidal* 3*–manifold M. Then either $\mathscr{F}$ contains a proper genuine sublamination, or $\mathscr{F}$ admits a transverse genuine lamination.*

In particular, for atoroidal 3–manifolds, essential laminations can be "promoted" to genuine laminations.

Motivating example: If M is closed, atoroidal and Haken, it either contains a surface with acylindrical complement, or it fibers over S^1. In the second case the fibration admits a transverse pseudo–Anosov flow X. After splitting open singular leaves of the stable and unstable singular foliation of X, the resulting laminations $\Lambda^{\pm}$ are genuine.

In the universal cover, the lifts of the laminations $\widetilde{\Lambda}^{\pm}$ are topologically of the form

$$\text{geodesic lamination of disk } \times \mathbb{R}$$

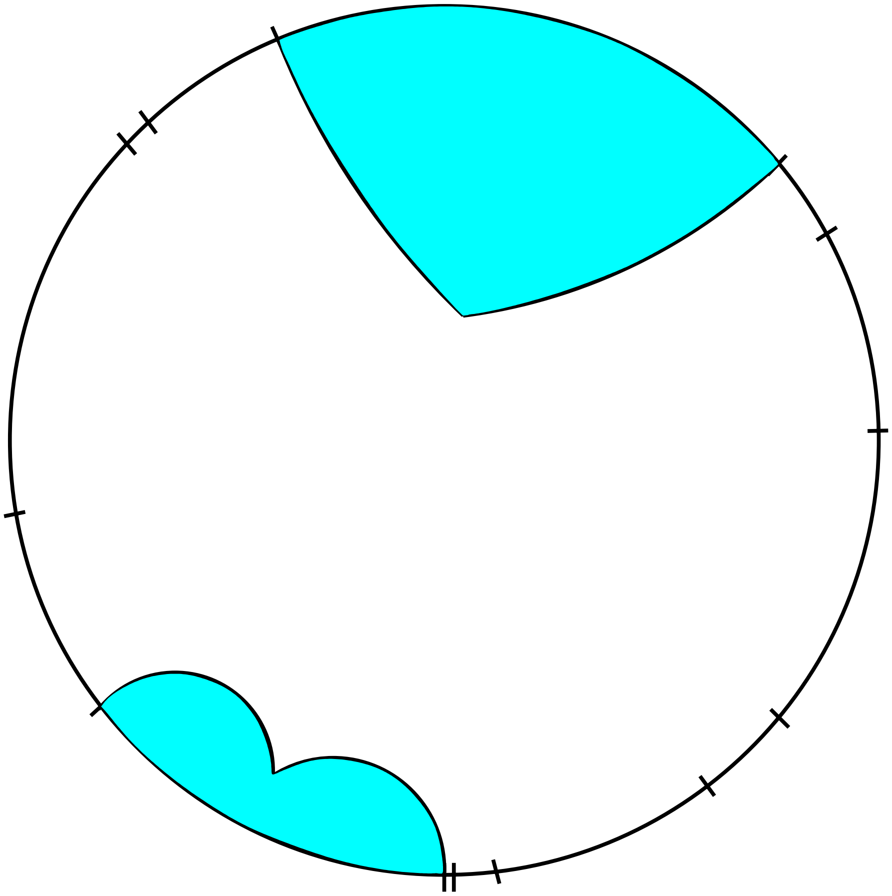


Ideas involved in the proof:

- By a blowing–down trick, we can assume $\mathscr{F}$ is *minimal* — that is, every leaf is dense.

- If M is atoroidal, Candel's theorem implies we can find a metric for which leaves are hyperbolic surfaces.

- Let L be the leaf space of $\widetilde{\mathscr{F}}$. Let E_∞ be the circle bundle over L, whose fiber over a leaf λ is $S^1_\infty(\lambda)$. Let $K \subset E_\infty$ be a closed, $\pi_1(M)$–invariant subset. Then either K intersects a.e. leaf in one point, or $K = E_\infty$. The proof of this uses Garnett's theorem on the existence of harmonic invariant measures.



For a leaf λ of $\widetilde{\mathscr{F}}$, consider the set $K \cap S^1_\infty(\lambda)$. We can define a function θ on λ by setting $\theta(p)$ equal to the biggest visual angle of a segment in $S^1_\infty(\lambda) - K$ as seen from p. This function is *leafwise (distributionally) subharmonic*, and globally bounded. By Garnett's theorem, it is actually harmonic.

- There is a dichotomy for the action of $\pi_1(M)$ on L: either the action is *compressible* in the sense that small intervals in L can be blown up to properly embedded rays or lines, or else there is an "invariant length scale" on L. In the second case, we say $\mathscr{F}$ arises from a *branched slithering* over S^1.

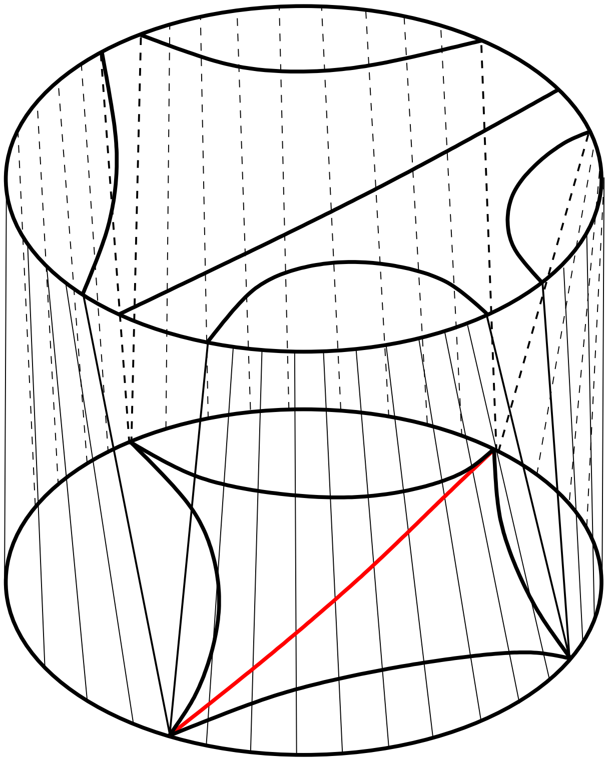
  In the second case, for each properly embedded positive (resp. negative) ray r in L, and each $p \in r$ there is a *minimal incompressible interval* $I(p, r) \subset r$ with lowest (resp. highest) point p. Define

$$\Phi(p, r) \to \text{highest point of I(p, r)}$$

The map Φ refines the partial ordering on L to points $p, q \in L$ with a common upper (resp. lower) bound, as follows: if $r(p)$ and $r(q)$ are rays containing p, q which eventually agree, there is an integer n such that $\Phi^n(p, r(p))$ and $\Phi^n(q, r(q))$ are comparable. Quotienting out by this relation is like zipping up a zipper.

- The asymptotic geometry of nearby leaves can be compared by *markers*. A marker is a map $H : I \times \mathbb{R}^+ \to \widetilde{M}$ which sends point $\times\, \mathbb{R}^+$ to leafwise geodesic rays, and $I \times$ point to transversals of length $\to 0$. The leafwise endpoints of a marker give rise to a collection of disjoint transversals in E_∞.

  If the action of $\pi_1(M)$ on L is compressible, limits of sequences of markers produce *long markers* which compare the asymptotic geometry of leaves arbitrarily far away in L. Even if $\mathscr{F}$ arises from a branched slithering, we can produce *unit markers* which project to segments of L of definite length, as measured from any given vantage point.

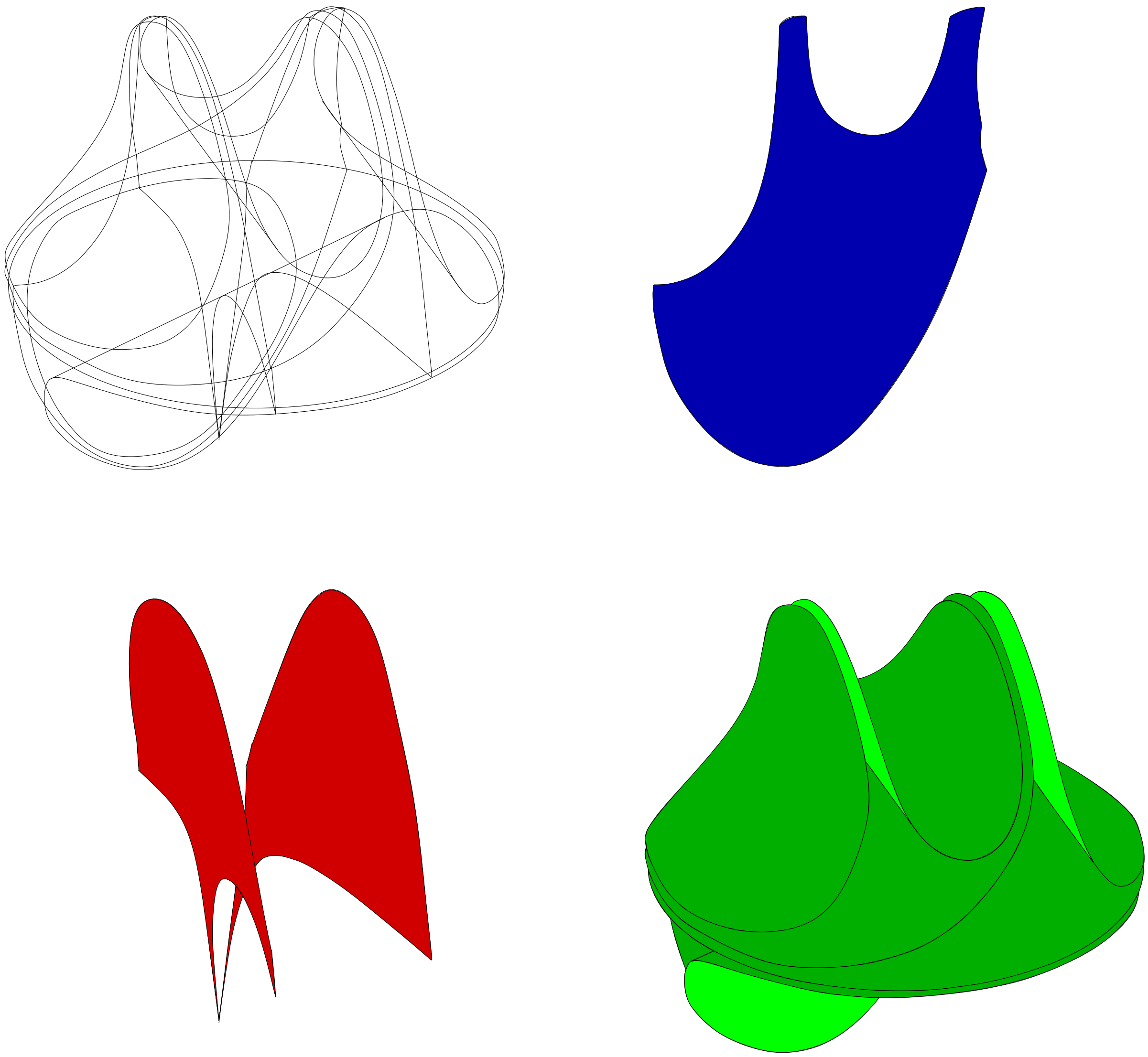


- Long or unit markers give rise to monotone relations between pairs of circles in E_∞. Subarcs of these circles which are quotiented by these relations are naturally nested, and their endpoints span leafwise geodesic laminations, which can be constructed from the combinatorics of L. The *uniform* bound on the transverse "length" (in L) of the markers controls the geometry of these leafwise laminations.

- The union of leafwise laminations sweeps out a branched lamination of $\widetilde{M}$. Since the branch locus is tangent to $\mathscr{F}$, this branched lamination can be equivariantly split open to a genuine lamination.

- In the case that $\mathscr{F}$ has one–sided branching or is $\mathbb{R}$–covered, there are different constructions of the laminations corresponding to the nonbranching directions. They detect the directions in which the geometry of the leaves is most distorted in the ambient manifold.

A “cutaway” picture showing how the pieces of $\widetilde{\Lambda}^{\pm}$ and $\widetilde{\mathscr{F}}$ interlock. Notice how the blue leaf branches in the positive direction, whereas the red leaves branch in the negative direction.



Corollary: *If M admits a taut foliation, then $\pi_1(M)$ either contains a $\mathbb{Z}\oplus\mathbb{Z}$ or $\pi_1(M)$ is word hyperbolic and the mapping class group of M is finite.*

Corollary: *If $\mathscr{F}$ is minimal and M is atoroidal, there is a transverse pseudo–Anosov flow.*

The leaf space of this transverse flow in $\widetilde{M}$ is homeomorphic to a disk D_{univ} in such a way that the action of $\pi_1(M)$ extends to the boundary circle S^1_{univ}. This circle is an example of a *universal circle* constructed by Thurston. In many circumstances (perhaps always?) the action of $\pi_1(M)$ on S^1_{univ} is *rigid* under deformations of $\mathscr{F}$, and there is a $\pi_1(M)$–equivariant sphere filling map from S^1_{univ} to $S^2_\infty(\widetilde{M})$, similar to the Cannon–Thurston map for M a surface bundle over a circle.