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Some Facts
Some Question's
Some Ideas
No Answers

Basics

(The Nature Of The Symplectic Category)

Monotonic Invariants

< 1985 (pre-Gromov)

$$(M,\omega) \rightsquigarrow \mathrm{vol}(M,\omega) = \int_M \omega^n$$

- $M \overset{s}{\hookrightarrow} N \Rightarrow \mathrm{vol}(M) \leq \mathrm{vol}(N)$
- $\mathrm{vol}(M,\alpha\omega) = |\alpha|^n \, \mathrm{vol}(M,\omega)$
- $\mathrm{vol}(B^{2n}(1),\omega) > 0$
- $\begin{cases} \mathrm{vol}(B^{2i}(1) \times \mathbb{R}^{2n-2i}) = \infty \\ 0 \leq i < n \end{cases}$
- $\mathrm{vol}(B^{2n}(1)) < \infty$

$\geq$ 1985

Symplectic capacities
(Gromov Gr - width)

- $M \overset{s}{\hookrightarrow} N \qquad c(M) \leq c(N)$
- $\to$ $c(M, \alpha\omega) = |\alpha| \, c(M, \omega)$
- $c(B^{2n}(1), \omega) > 0$
- $\to$ $\left\{ \begin{array}{l} c(B^{2i}(1) \times \mathbb{R}^{2n-2i}) < \infty \\ 0 < i \leq n \end{array} \right.$
- $c(\mathbb{R}^{2n}) = \infty$

Fact : $\exists$ ∞ independent Capacities

What about :

- $M \overset{s}{\hookrightarrow} N \qquad c(M) \leq c(N)$
- $c(M, \alpha\omega) = |\alpha|^k \, c(M,\omega)$
- $c(B^{2n}(1), \omega) > 0$
- $c(B^{2i}(1) \times \mathbb{R}^{2n-2i}) < \infty$

  $k \leq i \leq n$

  $c(B^{2i}(1) \times \mathbb{R}^{2n-2i}) = \infty$

  $0 \leq i < k$

Do these $2k$ - dimensional invariants $1 \leq k \leq n-1$ exist ?

How Unique are Symplectic Capacities?

$$
* \begin{cases}
\bullet \quad M \overset{s}{\hookrightarrow} N \qquad c(M) \le c(N) \\
\bullet \quad c(M, \alpha\omega) = |\alpha|\, c(M, \omega) \\
\bullet \quad c(B^{2n}) = c(B^2 \times \mathbb{R}^{2n-2}) = \pi
\end{cases}
$$

Contact ball: $B \subset \mathbb{R}^{2n}$
diffeomorphic to closed unit ball
+ contact type boundary.

$$\mathcal{B} = \{ B \}$$

Is $c|\mathcal{B}$ uniquely determined by $*$

$n=1$ trivial

$n=2$ seems already very difficult

C^0- symplectic maps and topology

Fix capacity c

$\rightsquigarrow$ Pseudo group of c-preserving homeomorphisms

Does S^4 admit an atlas consisting of homeomorphisms with c-preserving transition maps?

(Is there an intersting category of C^0- symplectic 4-MFLD's)

Is There A Piecewise Linear Symplectic Geometry?

$[x_0, \dots x_{2n}] \subset (\mathbb{R}^{2n}, \omega)$

simplex

(linear) affine invariants $\left(\omega(x_j - x_{i_0}, x_k - x_{i_0}) \right)_{j,k}$

A_{i_0} skew symmetric non-degenerate $2n \times 2n$ matrix

"$2n^2 - n$ numbers"

It would be nice to have an explicit formula for any symplectic capacity.

Fact (Gratza): Let $\Psi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the time-1-map of

$$\dot{x} = X_{H_t}(x)$$

$H : [0,1] \times \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ smooth and compact support.

$\Longrightarrow$ For given $\varepsilon > 0$ $\exists$ h_t, $t \in [0,1]$, continuous arc of homeomorphisms, $h_t(z) = z$ $\forall$ $|z| \geq R$, $h_0 = \mathrm{Id}$ with

- $h_t : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ is piecewise linear symplectic
- $|h_1(z) - \Psi(z)| < \varepsilon$ $\forall z \in \mathbb{R}^{2n}$

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Question: Given a compact symplectic manifold (M^{2n},ω) can it be triangulated so that the "cells" are symplectomorphic to linear simplices in $\mathbb{R}^{2n}$?

Fact: Consider the usual homology functor restricted to the category of symplectic $2n$-dim manifolds and symplectic embeddings.

Then H^* or H_* can be "enriched" to a functor carrying also symplectic information

$$F_k^{[a,b)}$$

$k \in \mathbb{Z}$ $\quad a \leq b \in \mathbb{R}$

$a \leq b \leq c \rightsquigarrow$ exact triangle

(symplectic homology for certain manifolds)

Note: Symplectic capacities can be explained using $F_*^{[r,\infty)}$

Modest and less ambitious question:

Is it possible to express the symplectic capacity $c(\mathcal{U})$, $\mathcal{U} \subset \mathbb{R}^{2n}$, bounded open set, as follows:

For $\varepsilon > 0$ take a triangulation $\mathcal{S}$ of a neighbourhood of $\mathcal{U}$ ~~by~~ by ε-small simplices. Produce by some 'algorithm' some number using the invariants of the simplices, so that

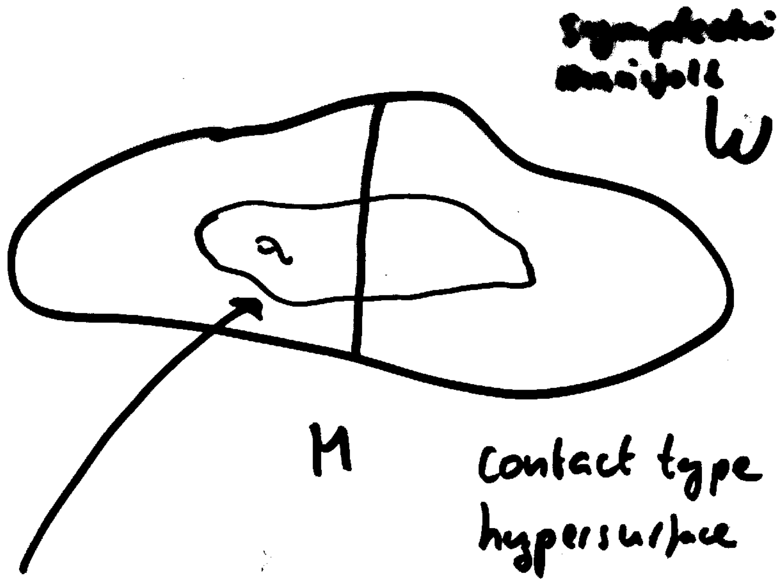
$$c(\mathcal{U}) = \lim_{\varepsilon \to 0} \inf_{\substack{\mathcal{S} \supset \mathcal{U} \\ \mathcal{S}\ \varepsilon\text{-fine}}} c_{\mathcal{S}}$$

11 ($\mathcal{U}$ sufficiently nice)

How can (pseudo-) holomorphic curves help understanding the situation?

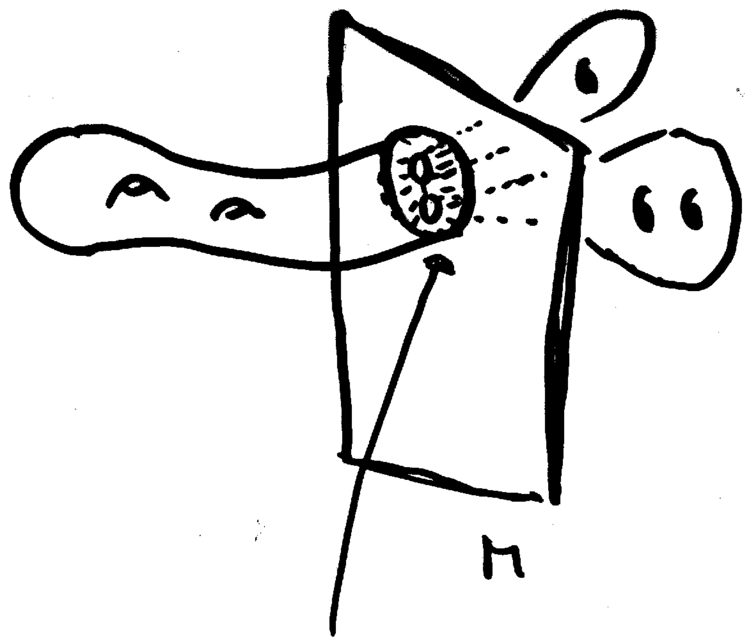
GR-WI & SFT & GR-WIDTH revisited

Gromov-Witten invariants
Symplectic Field theory use
Moduli spaces of holomorphic curves to define symplectic or contact invariants and relations between.

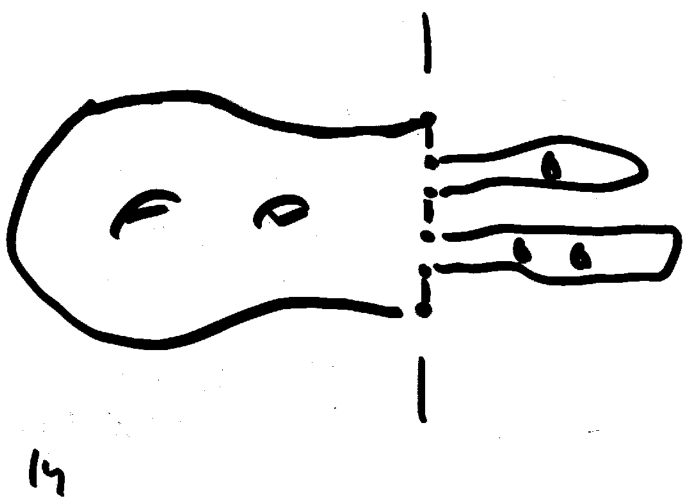


degenerate J (almost complex structure)
in a controlled way near M

Variation of "Stretching the neck"

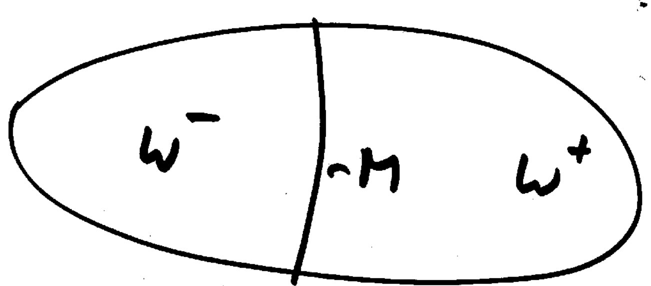


periodic orbits on M



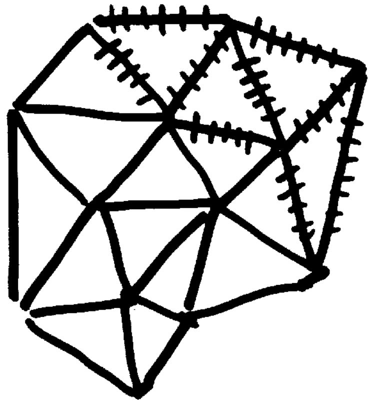
Goal: define suitable algebraic invariants counting holomorphic curves in the right, middle and left

W^- $\qquad$ M $\qquad$ W^+



Some condition on M is needed!

Idea



degenerate J in a suitable way along the $2n-1$ faces (the boundary of the simplex should be Levi flat for J), of course, there will be singularities

One should try (by choice of ∂) to have each simplex as a domain with flat $2n-1$ faces.

Good fact: There is a notion of a periodic orbit for boundaries of convex sets (interior $\neq \phi$) without assuming any smoothness!

Assume we degenerate

J_ε $\varepsilon \to 0$ and have

a sequence of holomorphic

curves u_ε with bounded

area:

In the limit one could expect pieces of holomorphic curves in some of the simplices and in some of their faces.

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The pieces in the simplices would be asymptotic to a periodic orbit on the boundary (so its area $\geq$

could quite possibly be the capacity of the simplex.

Wrap up

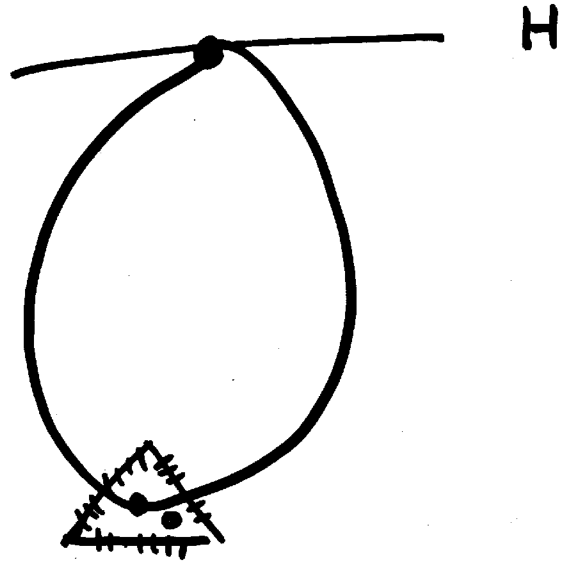
Assume symplectic manifolds admit symplectic triangulations then we perhaps could define some criteria of algebraic nature which by reversing the above procedure would give a holomorphic curve.

If we are very lucky we could obtain a symplectic theory without holomorphic curves.

However, we will - even if we are ultimately get rid of them - need them to get some ideas.

May be one should start here:

$$\mathbb{CP}^2 = \mathbb{C}^2 \cup H$$



Take suitable degenerating family of ∂'s and

20 see what happens.