Periodic Floer Homology –

a talk by

Michael Hutchings

based on joint work in progress with

Michael Thaddeus

at the

41st year Georgia Topology Conference

Papers available at

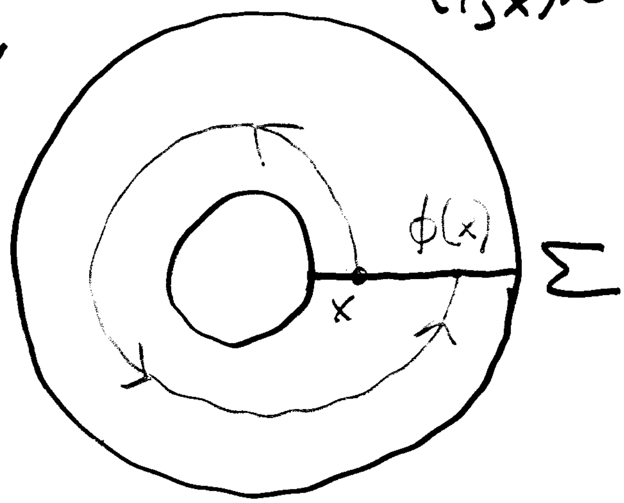
http://math.stanford.edu/~hutching

§1. Introduction

$$\phi : (\Sigma, \omega) \circlearrowleft$$

mapping torus $Y = \dfrac{[0,\overset{t}{\overset{\uparrow}{1}}] \times \Sigma}{(1,x) \sim (0, \phi(x))}$

Y



Natural flow ∂_t on Y.

A <u>periodic orbit</u> is an embedded closed orbit of this flow, in Y.

Goal: Given $h \in H_1(Y)$, we will define the periodic Floer homology $HP_*(\phi, h)$.

- chains generated by unions of periodic orbits w/ total homology class h.
- differential counts embedded (pseudo-) holomorphic curves in $\mathbb{R} \times Y$.

Motivation

- Taubes' "Seiberg-Witten = Gromov" theorem suggests that HP_* agrees with the Seiberg-Witten Floer homology of Y. But HP_* seems easier to understand and compute.

  (dynamics of ϕ.)

- HP_* is formally similar to, and leads to an interesting variant of, the contact homology of Eliashberg-Givental-Hofer.

<u>Other symplectic approaches to SWF homology</u>

- <u>Salamon:</u> SWF homology of mapping tori via symplectic Floer homology of certain induced maps on symmetric products of Σ. (This approach appears to be equivalent to ours via an "adiabatic limit" argument.)
- <u>Ozsváth-Szabó:</u> SWF homology of Heegard splittings via a version of Lagrangian intersections Floer homology in $S^d\Sigma$.

§2. Definition of Periodic Floer Homology

2A. ORBIT SETS.

Def An orbit set is a finite set of pairs $\{(\alpha_i, m_i)\}$, where

- The α_i's are distinct periodic orbits
- $m_i \in \{1, 2, \ldots\}$ "multiplicity"

$\alpha = \{(\alpha_i, m_i)\}$ is admissible if $m_i = 1$ when α_i is hyperbolic.

Notation

$$[\alpha] = \sum_i m_i [\alpha_i] \in H_1(Y).$$

$h \in H_1(Y)$, $\mathcal{A}(h) = \{\text{admissible } \alpha \mid [\alpha] = h\}$.

2B. FLOW LINES

$$\begin{array}{ccc} \Sigma \to Y & \qquad & \mathbb{R}^2 \to E = T_{vert}(Y) \\ \downarrow & & \downarrow \\ S^1 = \mathbb{R}/\mathbb{Z} & & Y \end{array}$$

Choose an almost complex structure $J : E \to E$, $J^2 = -I$.

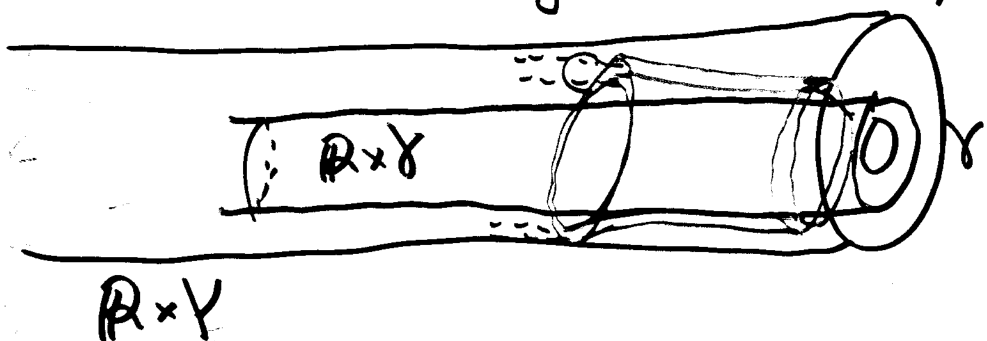
This extends to a unique a.c.s. J on $T(\mathbb{R} \times Y)$ satisfying

$$J(\partial_s) = \partial_t \qquad s = \mathbb{R}\text{-coord}$$

A <u>J-holomorphic curve</u> in $\mathbb{R} \times Y$ is a 2-dim submanifold $C \subset \mathbb{R} \times Y$ such that $J : TC \to TC$. <u>Example</u>: a fiber of $\Sigma \to \mathbb{R} \times Y \to \mathbb{R} \times S$

<u>Example</u> $C = \mathbb{R} \times \gamma$, where $\gamma \subset Y$ is a periodic orbit. (C = "trivial cylinder".)

More generally, a J-hol. curve $C \subset \mathbb{R}\times Y$ can have an end asymptotic to $\mathbb{R}\times\gamma^k$ as $s \to +\infty$ (an "<u>outgoing end</u> at γ of mult. k") or $s \to -\infty$ (an "<u>incoming end</u>" " " " " ")



<u>Def</u> A <u>flow line</u> from $\alpha = \{(\alpha_i, m_i)\}$ to $\beta = \{(\beta_j, n_j)\}$ is a J-hol. curve $C \subset \mathbb{R}\times Y$ such that:

- C has outgoing ends at α_i of total multiplicity m_i, incoming ends at β_j of total mult. n_j, and no other ends;
- C is embedded in $\mathbb{R}\times Y$, except possib for repeated trivial cylinders. (N.B: $[\alpha]=[\beta]$.)

2C. MONOTONICITY $h \in H_1(Y)$

Def ϕ is monotone w.r.t. h if

$$[\omega] = \lambda\big(c_1(E) + 2\,\mathrm{P.D.}(h)\big) \in H^2(Y;\mathbb{R}).$$

Lemma. Can achieve monotonicity by a symplectic isotopy $\iff h\cdot[\Sigma] \neq g(\Sigma)-1$.

- ϕ, ϕ' isotopic, monotone w.r.t. $h \Longrightarrow$ ϕ, ϕ' isotopic through symplectos monotone w.r.t. h

(Monotonicity holds in many natural examples.)

2D. THE CHAIN COMPLEX

Inputs: ϕ, h, J

Assumptions:

- ϕ, J suitably generic
- ϕ monotone w.r.t. h

Can drop this assumptio[n] using a Novikov ring

Def $\boxed{CP_*(\phi, h; J) = \mathbb{Z}/2\langle \mathcal{A}(h)\rangle}$

$\partial : CP_* \circlearrowleft$ (could orient and use $\mathbb{Z}$)

$\alpha \in \mathcal{A}(h)$, ($\mathbb{Z}/2$-graded by Lefschetz signs) (can refine)

$$\boxed{\partial\alpha = \sum_{\substack{\beta \in \mathcal{A}(h) \\ Z \in H_2(Y;\alpha,\beta) \\ I(\alpha,\beta;Z)=1}} \#\hat{\mathcal{M}}(\alpha,\beta;Z)\cdot\beta}$$

where • $H_2(Y;\alpha,\beta) = \left\{\begin{matrix}\text{rel. hom. classes of} \\ \text{flow lines from } \alpha \text{ to } \beta\end{matrix}\right\}$

(an affine space over $H_2(Y)$)

• $\hat{\mathcal{M}}(\alpha,\beta;Z)$ = moduli space of flow lines from α to β in rel. hom. class Z, modulo $\mathbb{R}$ action by s-translation

• $I(\alpha,\beta;Z)$ = relative index = expected dim. of moduli space

Thm ∂ is well defined, i.e.

$$\bigcup_{I(\alpha,\beta;Z)=1} \hat{\mathcal{M}}(\alpha,\beta;Z) \text{ is finite.}$$

Idea of proof Monotonicity implies that $\int_C \omega$ is constant for all C with $I(\alpha,\beta;Z)=1$, so we can use Gromov compactness. Can rule out standard technical problems:

- No "bubbling" because $\pi_2(Y)=0$. (if $g(\Sigma)>0$)
- No multiply covered curves arise, by a delicate index calculation. $\square$

"Thm" (not all details written down yet)

- $\partial^2 = 0$.
- $HP_*(\phi, h) = H_*(C_*(\phi, h; J), \partial)$ depends only on h and the smooth isotopy class of ϕ.

Idea of Proof: Modification of standard gluing arguments.

Note: it is essential that we require the orbit sets to be admissible. If not, then $\partial^2 \neq 0$, and in fact the euler characteristic would not be invariant.

So what is $HP_*(\phi, h)$? We will now see that it is computable, at least in simple examples.

§3. Examples!

3A. THE IDENTITY

(reduce to Morse theory on Σ)

$f : \Sigma \to \mathbb{R}$ Morse function

$\phi_t : \Sigma \circlearrowleft$ Hamiltonian flow of f

$\phi = \phi_1$

$Y \simeq S^1 \times \Sigma$

Take $h = d\,[S^1] \times [pt] \in H_1(Y)$.

Assume f "small" w.r.t. d.

$\mathrm{Crit}(f) \longrightarrow \mathrm{Fix}(\phi)$

index 1 $\longrightarrow$ hyperbolic

index 0, 2 $\longrightarrow$ elliptic

$$\mathcal{A}(h) = \left\{ \alpha : \mathrm{Crit}(f) \to \{0,1,2,\dots\} \;\middle|\; \begin{array}{l} \alpha(p) \le 1 \text{ if } \mathrm{ind}(p) = 1 \,; \\ \sum_{p \in \mathrm{Crit}(f)} \alpha(p) = d. \end{array} \right\}$$

Choose metric g on Σ so that the pair (f, g) is Morse-Smale $\rightsquigarrow$ J on Σ.

$$S^1 \times \Sigma \simeq Y$$

product a.c.s. $\longleftrightarrow$ $J(\partial_s + \nabla f) = \partial_t$

$$\left\{\begin{array}{c}\text{gradient flow lines}\\ \text{of } f \text{ (+ triv. cyls.)}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{flow lines of}\\ \text{periodic Floer hom.}\end{array}\right\}$$

Other flow lines don't contribute to ∂ due to S^1 symmetry.

<u>Conclusion</u>

S_d-invt. part of $\otimes^d$

$$(CP_*(\phi, h; J), \partial) = S^d\left(C_*^{\mathrm{Morse}}(f, g)\right) \otimes \mathbb{Z}/2$$

$$\Rightarrow \boxed{HP_*(id, h) = H_*(S^d \Sigma ; \mathbb{Z}/2)}$$

38. HYPERBOLIC TORUS AUTOMORPHISMS

$A \in SL_2\mathbb{Z}$, $\mathrm{spec}(A) \subset \mathbb{R} \setminus \{\pm 1\}$.

$\phi = \phi_A : T^2 = \mathbb{R}^2/\mathbb{Z}^2 \circlearrowleft$

<u>Thm</u> Differential in $HP_*(\phi, h)$ vanishes, so $HP_*(\phi, h)$ generated by unions of distinct periodic orbits w/ total hom. class h.

<u>Idea of Proof</u> Can use relative adjunction formulas to show that any flow line of relative index is a pair of pants, possibly together with trivial cylinders. Due to this and linearity of A, can use affine linear algebra to show that the count of pairs of pants is zero. $\square$

3C. DEGREE ONE CASE

If $h\cdot[\Sigma]=1$, then $HP_*(\phi,h)$ reduces to the ordinary symplectic Floer theory of ϕ.

- chains generated by $\mathrm{Fix}(\phi)$
- differential counts J-hol. sections of

$$\begin{tikzcd} \Sigma \arrow[r] & \mathbb{R}\times Y \arrow[d] \\ & \mathbb{R}\times S^1 \end{tikzcd}$$

Computations by Seidel, e.g. if $\phi =$ composition of Dehn twists around disjoint circles C_+, C_-, then

$$\bigoplus_{h\cdot[\Sigma]=1} HP_*(\phi,h) \simeq H_*(\Sigma\setminus C_{\pm}, C_{\pm})\otimes \mathbb{Z}/2$$

§4. Relations with other things

4A. SEIBERG-WITTEN FLOER HOMOLOGY

Y = closed oriented 3-mfld

$$\mathrm{Spin}^c(Y) = \{\text{spin-c structures on } Y\}$$
$$\approx H_1(Y)$$

$$c_1 : \mathrm{Spin}^c(Y) \to 2H^2(Y;\mathbb{Z})$$

If $\xi \in \mathrm{Spin}^c(Y)$ and $c_1(\xi)$ is not torsion, can define $SWF_*(Y,\xi)$, a $\mathbb{Z}/2$-graded $\mathbb{Z}/2$-module. (Can refine this...)

A mapping torus structure on Y induces a canonical isomorphism

$$\mathrm{Spin}^c(Y) = H_1(Y),$$

$$c_1(h) = c_1(E) + 2\,\mathrm{P.D.}(h).$$

<u>Conjecture</u> Let $Y =$ mapping torus of ϕ, $h \in H_1(Y)$. If $h \cdot [\Sigma] < g(\Sigma) - 1$, then

$$\boxed{HP_*(\phi, h) = SWF_*(Y, h) \otimes \mathbb{Z}_2}$$

<u>Reason for believing</u> The statement is a close analogue of Taubes' "SW = Gr" theorem, for the <u>noncompact</u> symplectic 4-manifold $(\mathbb{R} \times Y, \omega + ds \wedge dt)$.

It should have a similar proof, deforming the SW equations on $\mathbb{R} \times Y$ using ω.

The assumption $h \cdot [\Sigma] < g(\Sigma) - 1$ avoids a wall crossing during this deformation, but does not exclude most interesting cases.

<u>Thm</u> Conjecture true at euler characteristic level. $\square$

<u>Proof</u> Apply Taubes' thm to $S^1 \times Y$ ... $\square$

$$\chi(SWF_*(Y,h)) = SW(S^1\times Y, [S^1]\times h)$$

$$\overset{\text{Taubes}}{=} Gr(S^1\times Y, [S^1]\times h)$$

$$= \#\mathcal{A}(h)$$

$$= \chi(HP_*(\phi,h))$$

4B. COMPARISON WITH CONTACT HOMOLOGY

(analogy:)

PFH		Contact homology
Y = mapping torus of ϕ	$\longleftrightarrow$	Y = contact 3-mfld
dt	$\longleftrightarrow$	contact 1-form α
periodic orbits of ϕ	$\longleftrightarrow$	Reeb orbits of α
$(\mathbb{R}\times Y, \omega + ds\wedge dt)$	$\longleftrightarrow$	symplectization $(\mathbb{R}\times Y, d(e^s\alpha))$

(differences:)

flow lines embedded $\not\longleftrightarrow$ flow lines arbitrary

$\{(\alpha_i, m_i)\}$ $\not\longleftrightarrow$ count entire <u>partition</u> of m_i

only count total mult.

§5. Directions for future research

- Can define analogue of PFH for contact 3-mflds to get a new invt which might be interesting. (Examples?)
- Can try to define "quantum product"

$$HP_*(\phi) \otimes HP_*(\psi) \longrightarrow HP_*(\phi \circ \psi)$$

by counting J-hol. curves in bundle

$$\begin{array}{c} \Sigma \longrightarrow X \\ \downarrow \\ S^2 \setminus \{0,1,\infty\} \end{array}$$

with monodromy $\phi, \psi, \phi\circ\psi$ around punctures.

Should lead to formula for SW invts of symplectic Lefschetz pencils in terms of quantum products.

$$h \mapsto -2c_1(E) - h$$

deg $2g-2$