Gromov-Witten Invariants and Symplectic Sums

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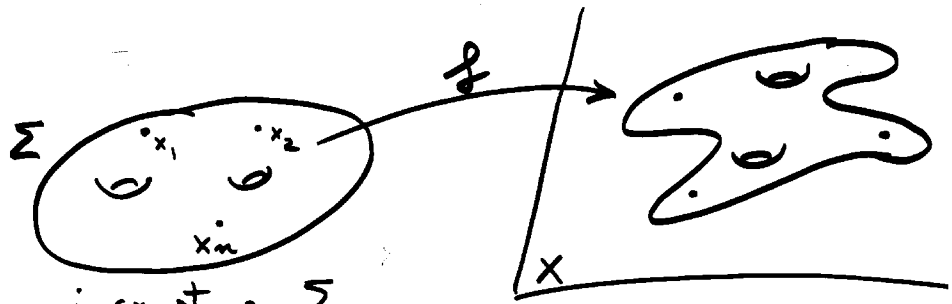
based on joint work with Tom Parker

0. Review of GW Invar.

(X, ω) sympl, J alm. cx. str



j cx str on Σ

genus g, $A \in H_2(X)$

moduli space of "holo" curves

$$\mathcal{M}_{g,n}(X, A) = \{ (f, j, x_1, \ldots x_n) \,/\, f : \Sigma \to X \text{ (alm) holo },\ [f] = A \} / \mathrm{Diff}(\Sigma)$$

gen. orbifold of dim

$$2 c_1(X)[A] + (\dim X - 6)(1-g) + 2n$$

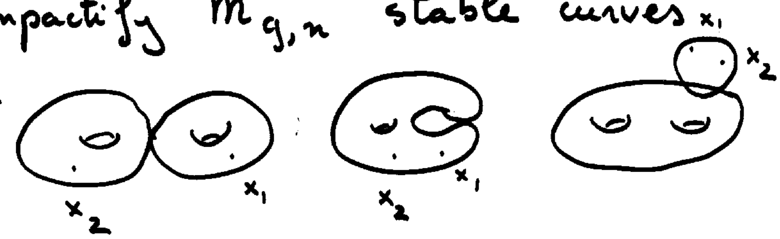
$$\mathcal{M}_{g,n}(X,A) \xrightarrow{st \times ev} \mathcal{M}_{g,n} \times X^n$$

$$[f, j, x_1, \dots, x_n] \longmapsto ([j, x_1, \dots, x_n], f(x_1), \dots, f(x_n))$$

$\mathcal{M}_{g,n}$ = moduli space of cx. str on Σ

compactify $\overline{\mathcal{M}}_{g,n}$ stable curves

eg:



boundary strata of codim$_{\mathbb{C}}$ = # nodes

$$\overline{\mathcal{M}}_{g,n}(X,A) \xrightarrow{st \times ev} \overline{\mathcal{M}}_{g,n} \times X^n$$

$GW_{X,A,g,n}$ = homology class of image

invariant - indep of J alm cx str
- indep of ω in a defm. class of sympl str
- preserved by symplectomorph

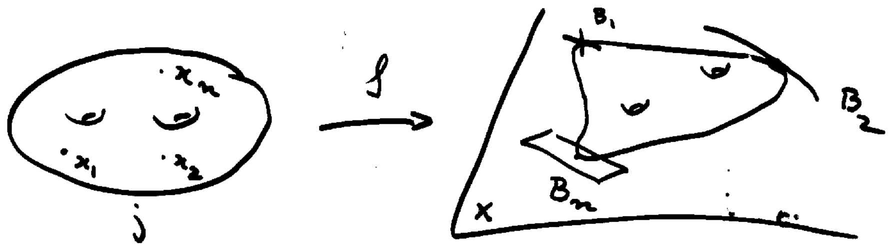
If we want numbers:

Given $\beta_1, \dots, \beta_n \in H_*(X)$

$\kappa_1, \dots, \kappa_n \in H_*(\overline{\mathcal{M}}_{g,n})$

choose geom repr $B_1, \dots, B_n$

$K_1, \dots, K_n$



$GW_{X,A,g,n}(\beta_1, \dots, \beta_n; \kappa_1, \dots, \kappa_n) =$

$= \#$ "holo" curves $[f, j, x_1, \dots x_n]$

s.t. $f(x_i) \in B_i$

$[j, x_1 \dots x_n] \in K_1 \cap \dots \cap K_n$

(indep of choice of geom. repr.)

generating function

$$GW_X = \sum_{A,g,n} GW_{X,A,g,n} \cdot \lambda^{2g-2} \cdot e^{A} \cdot \frac{u^n}{n!}$$

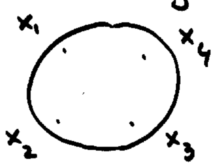
$GT_X = \exp(GW_X)$ counts possibly disconnected curves

Q: How to compute it?

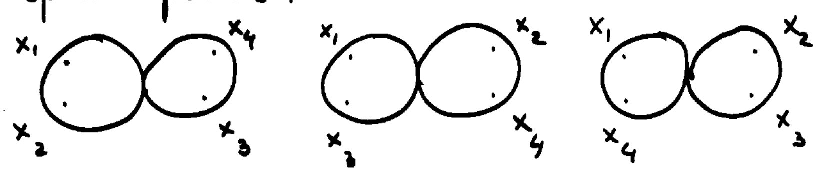
(a) Pull back rel in $H^*(\overline{\mathcal{M}}_{g,n})$

eg: $\overline{\mathcal{M}}_{0,4} \simeq \mathbb{CP}^1$ (by cross ratio)

general point :



"Special" points :



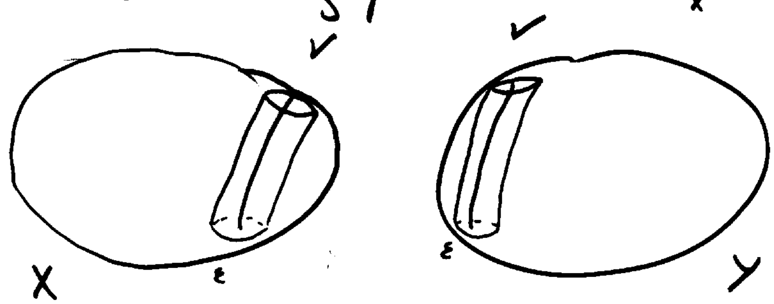
$\Rightarrow$ $g=0$ GW invar of $\mathbb{CP}^N$ (Kontsevich, Ruan-Tian)

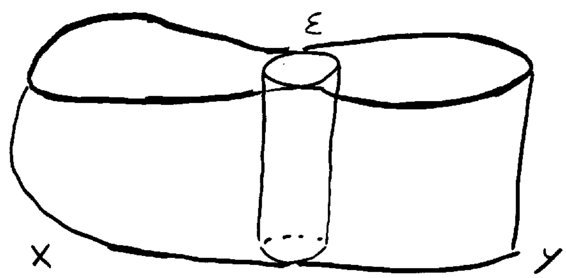
Keel : complete description of $H^*(\overline{\mathcal{M}}_{0,n})$

(b) split the target - Sympl. Sum

let X, Y sympl

V codim 2 sympl embed. st. $N_X V \simeq (N_Y V)^*$

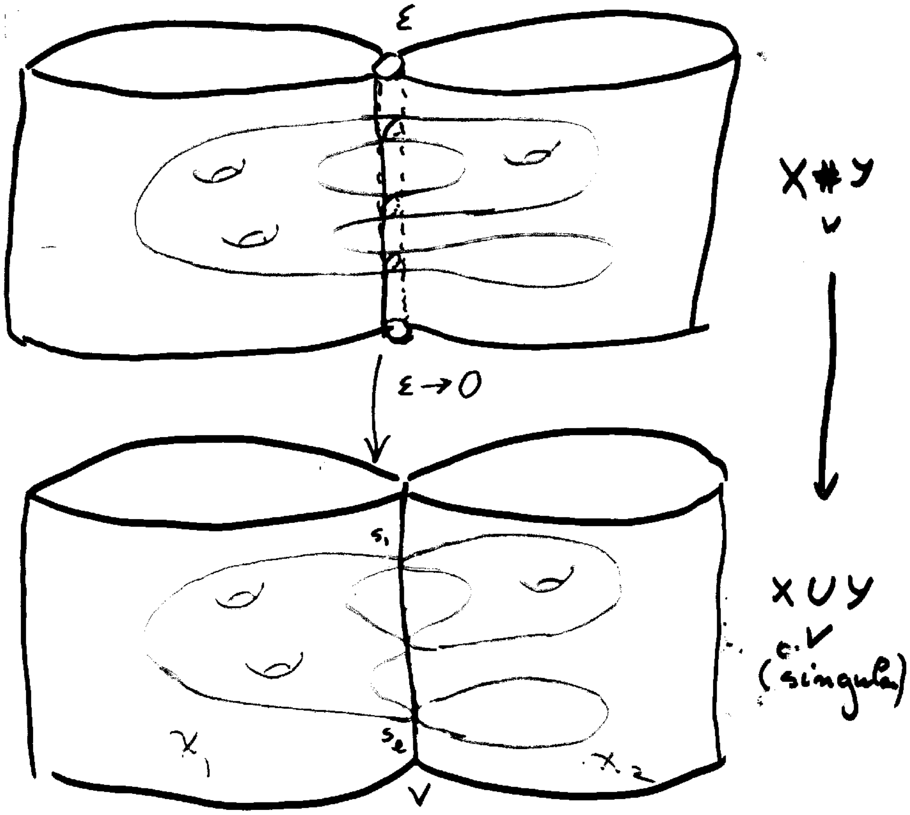




$X \#_V Y$ sympl. sum along V

in fact: fam of sympl. sums, depending on $\varepsilon > 0$, all defm. equiv.

holo curves:



$f(z) = a z^{s}$ $\quad$ s = multiple of int.

$$\chi = \chi_1 + \chi_2 - 2\ell$$

1. suppose no comp. land in V
in the limit we see "holo" curves
in X and Y with matching inter-
sections with V (including multipl.)

- define relative GW invar GW_X^V that
  counts "holo" curves which intersect V
  in prescribed multipl. $s_1, \dots, s_\ell$

- two such curves in X and Y which
  match along V glue to give
  $s_1 \cdots s_\ell$ "holo" curves in $X \#_V Y$

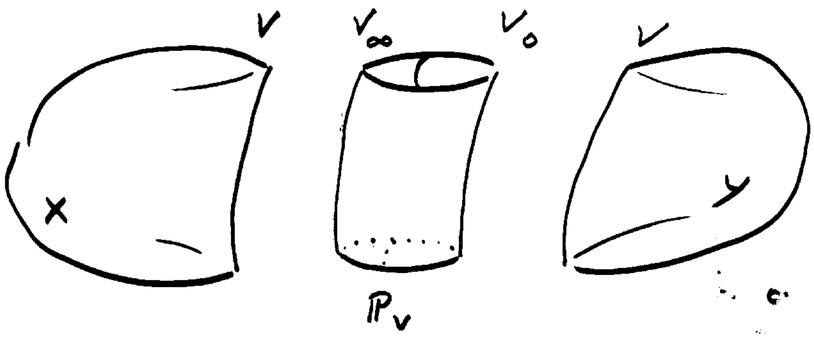
- so under assumption above,

$$GT_{X \#_V Y} = GT_X^V * GT_Y^V$$

Thm (I-Parker)

$$GT_{X\#_V Y} = GT_X^V * (GT_{\mathbb{P}_V}^{V_\infty, V_0})^{-1} * GT_Y^V$$

where $\mathbb{P}_V = \mathbb{P}(N_X V \oplus \mathbb{C})$ and V_0, V_∞ are the 0 and ∞ sections

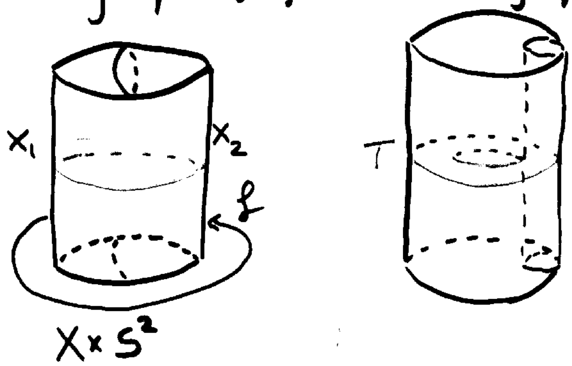


similar formulas were obtained by Li-Ruan and Eliasberg-Hofer

eg 0 : GW_X invar for $A=0$, $g=1$, fixed j on T^2 is equal to $\chi(X)$; in this case formula becomes $\chi(X \#_V Y) = \chi(X) - 2\chi(V) + \chi(Y)$

eg 1. Symplectic Mapping Tori

X sympl, $f: X \to X$ sympl. morph.



X_f sympl mapping torus

$X_f = \underset{f(x_1)=x_2}{\#} X \times S^2$

p fixed pt of f $\rightsquigarrow$ torus T in X_f

let N_d = GW invar of X_f in $A = d[T]$, $g=1$

$$F(t) = \sum_{d=0}^{\infty} N_d \cdot t^d \quad \text{generating fcn}$$

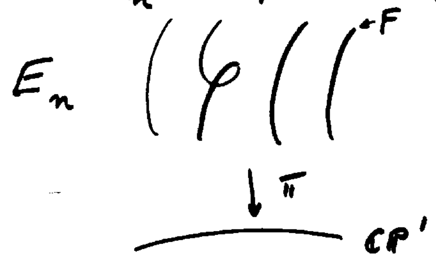
then $F(t) = \zeta_f(t)$ zeta fcn of f

eg. K fibered Knot in S^3

monodromy $\rightsquigarrow f: \Sigma \to \Sigma \rightsquigarrow \Sigma_f$

$$F(t) = \frac{A_K(t)}{(1-t)^2}$$ $\leftarrow$ Alexander poly

let E_n = elliptic surface



generic fiber T^2
12 n nodal fibers

$$E_n(K) = E_n \underset{F=T}{\#} \Sigma_g \quad \text{sympl}$$

homeo to E_n

but $g=1$ GW invar above

$$F(t) = (1-t)^{n-2} \cdot A_K(t)$$

$\rightsquigarrow$ exotic sympl str on E_n

(Fintushel Stern examples)

$E_n(K) \times S^2$ are diffeo to $E_n \times S^2$

but not sympl. defm. equiv.

(previous such eg : McDuff , Ruan)

eg 2. Relations in $H^*(\overline{\mathcal{M}}_{g,n})$

$\overline{\mathcal{M}}_{g,n}$ cx orbifold of $\dim_{\mathbb{C}} = 3g-3+n$

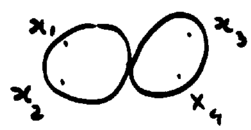
$L_i \downarrow \overline{\mathcal{M}}_{g,n}$ line bd whose fiber at $(\Sigma, x_1 .. x_n)$ is $T^*_{x_i}\Sigma$

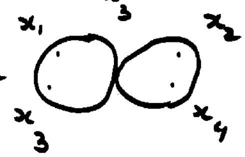
let $\varphi_i = c_1(L_i) \in H^2(\overline{\mathcal{M}}_{g,n})$ descendant

$\kappa_a = \pi_*(\varphi_{n+1}^{a+1}) \in H^{2a}(\overline{\mathcal{M}}_{g,n})$ tautological

where $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ forgets last pt.

eg $g=0$ $\quad PD(\varphi_1) =$  in $H_2(\overline{\mathcal{M}}_{0,n})$

implies 

$=$ 

Thm (I) Any degree g poly in descendant and tautological classes vanishes on $H^*(\mathcal{M}_{g,n})$

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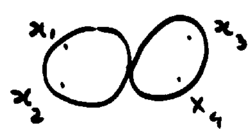
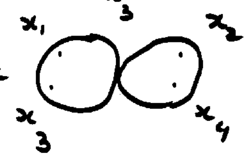
implies 

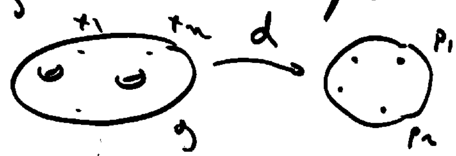
= 

Thm (I) Any degree g poly in descendant and tautological classes vanishes on $H^*(\mathcal{M}_{g,n})$

extends results of Loojenga ($n=0$)
and Getzler ($g=2$)

Key idea: Take $X = S^2$, $V = \{p_1, \dots, p_r\}$

$\overline{\mathcal{X}}_{d,g}$ = degree d holo maps



with certain prescribed branching
pattern at points $p_1 \cdots p_r$

$$\begin{tikzcd} & \overline{\mathcal{X}}_{d,g} \arrow[dl, "st"', "\text{domain}"] \arrow[dr, "e", "\text{target}"'] & \\ \overline{\mathcal{M}}_{g,n} & & \overline{\mathcal{M}}_{0,r} \end{tikzcd}$$

take rel in $H^*(\overline{\mathcal{M}}_{0,r})$ pull back
and push forward to get rel in $H^*(\overline{\mathcal{M}}_{g,n})$