Mapping Class Groups

vs.

Arithmetic and Linear Groups

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Part I: Arithmetic Groups

$$
\begin{array}{ccc}
 & SL_2(\mathbb{Z}) & = \pi_0(\mathrm{Diff}^+(S^1\times S^1)) \\
\swarrow & & \downarrow \\
SL_n(\mathbb{Z}),\ Sp_{2n}(\mathbb{Z}) & & \pi_0(\mathrm{Diff}^+(S)) \\
\downarrow & & \| \\
\text{Arithmetic Groups} & & \mathrm{Mod}_S \text{ or } \Gamma_{g,b}
\end{array}
$$

rank $= 1$	rank ≥ 2
(Act on a hyperbolic space)	($SL_n(\mathbb{Z})$ with $n \geq 3$)

(g = genus
b = # boundary components)

Question (W. Harvey): are $\Gamma_{g,b}$ arithmetic?

Answer (I.): No.

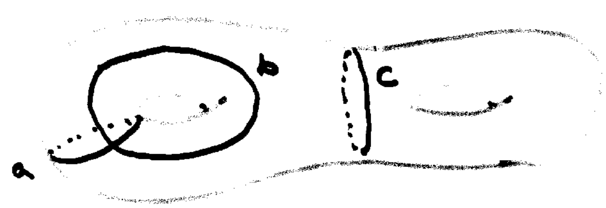
<u>First proof.</u> Suppose Γ is arithmetic and $\Gamma \approx \Gamma_{g,b}$.

Case 1: $\operatorname{rank} \Gamma = 1$.

Using the action on a hyperbolic space, one can see that

$$\Gamma \not\supset (\mathbb{Z} * \mathbb{Z}) \times \mathbb{Z} .$$

$$\Gamma_{g,b} \supset (\mathbb{Z} * \mathbb{Z}) \times \mathbb{Z} = \langle t_a^2, t_b^2, t_c \rangle$$



Case 2. rank $\Gamma = 2$

By Margulis, for every

$$1 \to K \to \Gamma \to \Gamma' \to 1$$

either K is central or Γ' is finite.

$$1 \to \text{Torelli} \to \Gamma_{g,b} \to Sp_{2g}(\mathbb{Z}) \to 1$$

Torelli $\supset$ {twists about separating circles}

$Sp_{2g}(\mathbb{Z}) = \mathrm{Aut}(H_1(\cdot), \langle \cdot, \cdot \rangle)$

$\square$

This proof tells us nothing interesting about MCG.

1 Why to ask this question?

Γ acts on G/K $(\approx \mathbb{R}^n)$ cocompactly or with finite covolume

$\Gamma_{g,b}$ acts on $T_{g,b}$ – Teichmüller space

$\mathrm{vol}(\Gamma_{g,b}\backslash T_{g,b}) < \infty$ (H. Masur)

There is a natural boundary of $\Gamma\backslash G/K$ (Borel-Serre)

There is a natural boundary of $\Gamma_{g,b}\backslash T_{g,b}$ (Harvey, I. details)

Computation of

$\mathrm{vcd}\,\Gamma$ (Borel-Serre)

$\mathrm{vcd}\,\Gamma_{g,b}$ (J. Harer)

Computation of

$\chi(\Gamma)$

(G. Harder)

Products of values of $\zeta(\cdot)$ at integer points

Mostow rigidity

$\chi(\Gamma_{g,b})$

(Harer-Zagier, Penner)

$\chi(\Gamma_{g,1}) = \zeta(1-2g)$

Royden's Theorem

$\mathrm{Iso}(T_{g,b}) = \Gamma_{g,b}$ with respect to the Teichmüller metric —

— Proof by I.

(Royden's proof is very different)

Notice that W. Harvey asked his question and suggested the analogy <u>before</u> all these results were obtained.

Second proof

For an abstract group H one can define <u>rank</u> H as follows (Prasad - Ragunathan - Ballmann-Eberlein)

Let $(H)_i := \{h \in H : \mathrm{cent}(h) \supset \mathbb{Z}^i$ as a subgroup of finite index $\}$

$\mathrm{rank}'(H) = \min i$ such that

$$H = h_1 (H)_i \cup \dots \cup h_N (H)_i$$

$\operatorname{rank} H = \max \operatorname{rank}' H'$
$[H:H'] < \infty$

(technical step to deal with torsion)

<u>Theorem (I).</u> $\operatorname{rank}(\Gamma_{g,b}) = 1$.

<u>Ingredients of the proof:</u>

- Centralizers of pseudo-Anosov's are virtually $\mathbb{Z}$
- Compactness of Thurston's boundary of Teichmüller space
- A theorem of Fathi.

$t_\alpha^n f$ is p. A. for $\forall n$ with $\exists$ possible

(α, f) is "generic" $\Uparrow$

<u>Theorem</u> (Prasad - Ragunathan)
rank Γ = rank G = rank G/K
for arithmetic Γ.

(uses compactness of G in Zariski topology).

<u>End of the second proof</u>:
If $\Gamma \approx \Gamma_{g,b}$, then rank $G/K = 1$. Use the easy (rank $=1$) part of the first proof. $\square$

<u>Remark</u> In the computation of vcd $\Gamma_{g,b}$ and in the proof of Royden's theorem (by I.), $\Gamma_{g,b}$ behaves like rank ≥ 2 group, in contrast with the above.

<u>Third proof.</u>

Abstract Commensurators:

$$\mathrm{Comm}(H) = \{ \varphi : H_1 \to H_2 \mid [H:H_i] < \infty,\ \varphi \text{ is an iso} \} / \sim$$

$\varphi_1 \sim \varphi_2$ iff $\varphi_1 | H' = \varphi_2 | H'$.
for some H', $[H : H'] < \infty$

Clearly $H \to \mathrm{Comm}(H)$
by $h \mapsto i_h$, $i_h(g) = hgh^{-1}$.
Usually, this map is <u>injective</u>

<u>Theorem</u> (Borel) If Γ is arithmetic, then
$[\mathrm{Comm}(\Gamma) : \Gamma] = \infty$

<u>Idea of the proof.</u> For $SL_n(\mathbb{Z})$, every element of $SL_n(\mathbb{Q})$ induces an element of $\mathrm{Comm}(SL_n(\mathbb{Z}))$ (conjugation)

$\square$

1. **Remark**. By a theorem of Margulis, if Γ is a lattice, then Γ is arithmetic iff $[\mathrm{Comm}(\Gamma):\Gamma]=\infty$ (independently of rank).

Theorem (I.) $\mathrm{Comm}(\Gamma_{g,b}) = \overline{\Gamma_{g,b}}$ (include orientation-reversing mapping classes).

$$\Uparrow$$

Theorem (I.) If H_1, H_2 have a finite index in $\Gamma_{g,b}$ and $g: H_1 \to H_2$ is an iso, then g is a conjugation by $\in \overline{\Gamma_{g,b}}$.

$\Uparrow$

Complex of curves $C(S)$
(W. Harvey)

vertices: isotopy classes of nontrivial circles on S

Simplices: collections of (isotopy classes) of <u>disjoint</u> circles

One may consider also just 1-skeleton (a graph). It contains the same information.

12-A

Theorem (I. for $g \geq 2$,
M. Korkmaz for $g = 1, 0$).
If $S = S_{g,b}$, then

$$\operatorname{Aut} C(S) = \overline{\Gamma}_{g,b}.$$

Secret: how it was discovered?

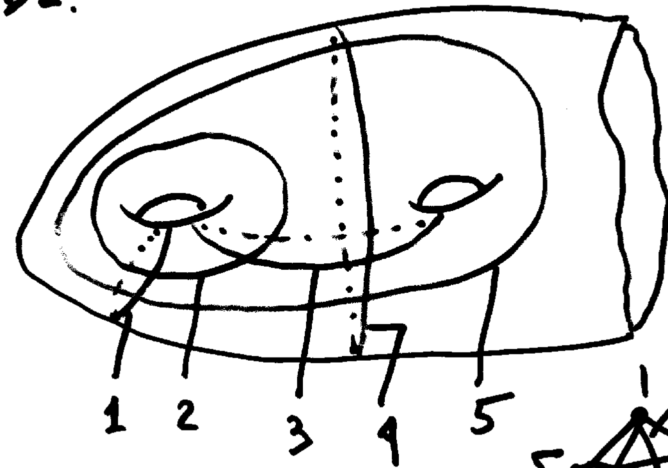
1. $\operatorname{Aut} C(S) = ?$
2. $\operatorname{Aut} C(S) = \overline{\Gamma}_{g,b} \Longrightarrow$
   $\operatorname{Out}(\Gamma') = \operatorname{Aut}(\Gamma') / \operatorname{Inn}(\Gamma')$
   is finite for every Γ',
   $[\overline{\Gamma}_{g,b} : \Gamma'] < \infty$.
   Similar property holds for arithmetic groups by Mostow.
3. $\operatorname{Aut} C(S) = \overline{\Gamma}_{g,b} \Longrightarrow$ (along the lines of Mostow Th)
   Royden's theorem.

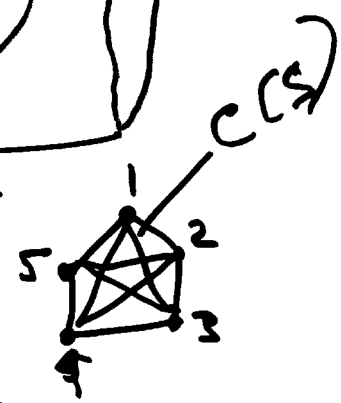
4. Automorphisms of $C(S)$ preserve the "geometric intersection number $= 1$" property.

5. The rest are technical details (not quite, of course ...), since you believe in the theorem now.

13-A. Proof of 4.

$g \geq 2$.

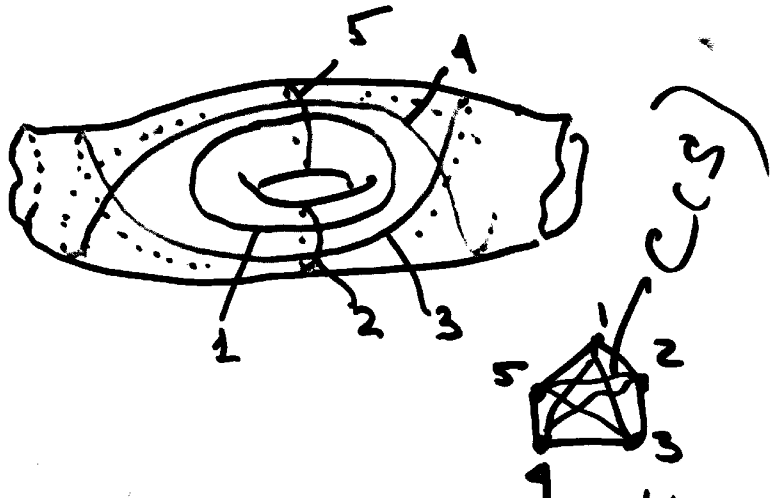




$i(\alpha_1, \alpha_2) = 1 \iff \exists\, \alpha_3, \alpha_4, \alpha_5$ forming a "pentagon" and such that α_4 divides S into a torus with one hole containing α_1, α_2 and the rest.

B-B

g=1

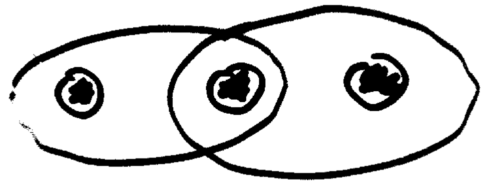


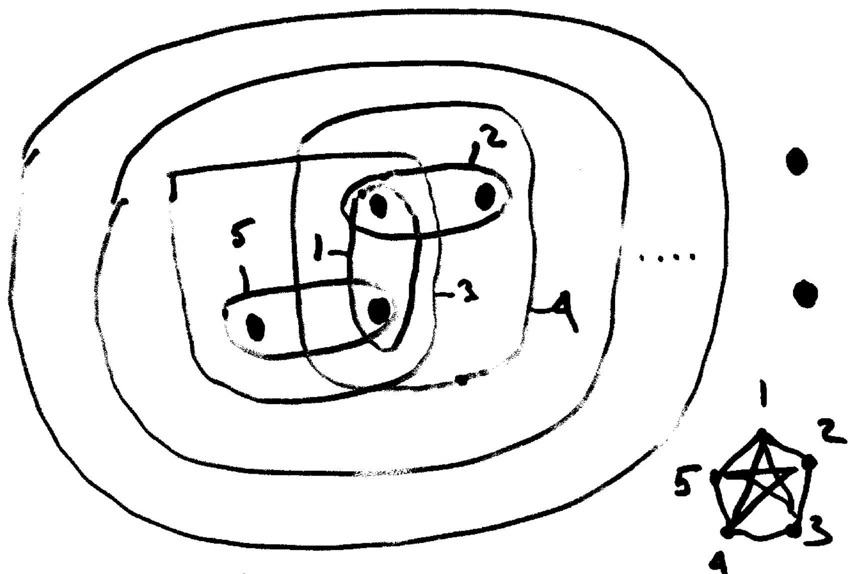
$\alpha_1, \alpha_2, \alpha_3$ are nonseparating

α_4, α_5 cut off a torus with one hole.

13-C $g=0$. No intersection number!

Instead: simple pairs





(There is a later proof by F. Luo,
(for all cases), using different language
but similar ideas)

14. <u>Remark</u>.

$C(S)$ is an analogue of the <u>Tits buildings</u> associated to Lie groups G.

$\operatorname{Aut} C(S) = \Gamma_{g,b}$ is an analogue of Tits' theorem about automorphisms of buildings.

The Tits' theorem, in turn, is an analogue of the Basic Theorem of Projective Geometry.

Here $\Gamma_{g,b}$ behaves like a group of <u>rank ≥ 2</u> (because $\dim C(S) > 0$).

Part II : Linear Groups

Theorem (Birman - Lubotzky - McCarthy)
Every virtually solvable subgroup of $\Gamma_{g,b}$ is virtually abelian.

Theorem (I., McCarthy ; independently)
If $\Gamma \subset \Gamma_{g,b}$, then either $\Gamma \supset \mathbb{Z} * \mathbb{Z}$, or Γ is virtually solvable

For linear groups: Tits
(Tits Alternative)

16. <u>Theorem</u> (I.) Every $\Gamma < \Gamma_{g,b}$ is either virtually solvable, or contains a maximal subgroup of infinite index.

For linear groups (Margulis–Soifer)

<u>Theorem</u> (I.) The intersection of all maximal subgroups of a $\Gamma \subset \Gamma_{g,b}$ is [redacted] nilpotent.
(It is the so-called Frattini subgroup of Γ).

For linear groups : Platonov

17. Question: Is the same true for the intersection of all <u>finite index</u> maximal subgroups?
For linear groups, Platonov's proof gives exactly this. (His proof uses the theory of algebraic groups and the <u>Dirichlet theorem about primes</u> in <u>arithmetic progressions</u>.)

18 A more special property of Linear Groups.

Def. Let G be a reasonably big group (not virtually solvable is sufficient). Then $\langle G \times G, t \mid t(g,1)t^{-1} = (g,g) \ \forall g \in G \rangle$ is a Formanek-Procesi group.

<u>Theorem</u> (F.-P.) A linear group cannot contain a Formanek-Procesi group.

<u>Corollary</u> (F.-P.) $\mathrm{Aut}(\underbrace{\mathbb{Z} * \dots * \mathbb{Z}}_{n \geq 3})$ is not linear

13. <u>Observation</u> (T. Brendle, H. Hamidi-Tehrani)

The Formanek-Procesi construction of F.-P. subgroups of $\mathrm{Aut}(\mathbb{Z} * \ldots * \mathbb{Z})$ makes sense for $\Gamma_{g,b}$.

<u>Theorem.</u> (B. & H.-T.)

It does not lead to <u>any</u> F.-P. subgroup of $\Gamma_{g,b}$.

<u>Theorem</u> (I.) $\overline{\Gamma_{g,b}}$ contains no F.-P. subgroups.

20 The key tool:

<u>Theorem</u> (I.) Every <u>irreducible</u> subgroup of $\Gamma_{g,b}$ contains a pseudo-Anosov element, and even two independent pseudo-Anosov elements (in particular, $\mathbb{Z} * \mathbb{Z}$ generated by two pseudo-Anosovs), if not virtually $\simeq \mathbb{Z}$.