LEGENDRIAN KNOTS PROBLEM SESSION. (GEORGIA TOPOLOGY CONFERENCE, MAY 31, 2001.)

Conducted by John Etnyre. For the most part, we restricted ourselves to problems about Legendrian knots in dimension three. Unless otherwise stated, the ambient contact manifold is assumed to be (S^3, ξ_{std}). tb is the Thurston-Bennequin invariant and r is the rotation number.

1. Examples

We "understand", i.e, can completely classify, the following:

1. The unknot in any tight contact manifold. Strangely enough, we however do not understand unknots in overtwisted contact manifolds.
2. Torus knots in any tight contact manifold.
3. The figure 8 knot in (S^3, ξ_{std}). (The figure 8 knot has a unique $tb = -3$ representative, and all others are stabilizations of this unique maximal tb representative.)
4. Connect sums $K_1 \# K_2$, provided we understand both K_1 and K_2.
5. Cables of knots we already understand.
6. The previous two combined gives us all the zero entropy knots (those obtained by iterated connect summing and cabling operations).
7. Legendrian 5_2 knots, i.e., the Chekanov examples?? Maybe....

Define the positive and negative stabilizations $K \mapsto S_+(K)$ and $K \mapsto S_-(K)$ by adding zigzags to K to decrease tb by 1 and increase or decrease r by 1. We then have the following theorem of Fuchs-Tabachnikov: If K and K' are topologically isotopic, then $S_+^{n_1} S_-^{n_2}(K)$ and $S_+^{n_1} S_-^{n_2}(K')$ are Legendrian isotopic.

Note: The positive stable classification of Legendrian knots in a knot type is equivalent to the classification of transversal knots for the same topological knot type.

Need to understand:

1. Fibered knots (both pseudo-Anosov and not pseudo-Anosov — some of them are zero entropy knots from above, however).
2. Hyperbolic knots (very important)! For example, 8_{20} has interesting holonomy and is fibered.
3. Knots in overtwisted contact structures.

Can we generalize the Chekanov examples to other twisted Whitehead doubles of the unknot? (Historical remark: The "Chekanov examples" are originally due to Gompf.) The rationale behind the examples is the following: We are Whitehead doubling Legendrian unknots with the same tb and different r, and the new tb is $+1$, whereas the new r is 0. Therefore, the doubled Legendrian knots should have the same tb and r but should be nonisotopic, since, morally speaking, the underlying core curves are distinct.

Problem 1. *Distinguish the 5_2 examples using convex surface theory (3-dimensional methods). Can you distinguish the core Legendrian unknots?*

Problem 2 (Main question for transversal knots)**.** *Are there nonisotopic transversal knots with the same self-linking number?*

Problem 3. *Compute the maximal tb for 9_{42}. There exists a $tb = -15$ example which is not a stabilization, but the best known bound is $tb = -14$. This indicates the possibility of two Legendrian knots which are not stabilizations but which have different tb's.*

Note: Bounds are obtained using the Bennequin inequality, and the Kaufmann polynomial, HOMFLY polynomial, etc. The bound given in the above problem is the best we can do using current technology.

Try to do computer-assisted calculations.

2. Structural theorems

For example, the results on connect sums and cables above are structural theorems.

What about:

1. Whitehead doubles and other satellite constructions?
2. Murasugi sum?
3. Crossing changes? (For example, is there a formula for max tb changes under crossing changes?)
4. Other favorite topological abuse of knots.
5. If L_1 and L_2 are topologically isotopic with the same tb, is there a set of moves taking you from L_1 to L_2?

For example, a basic move suggested by Lenny Ng is given in 1. However, this is not sufficient.

Question 1. *Can Legendrian knots with nonmaximal tb always be destabilized? (Etnyre-Ng conjecture: No.)*

Note: All known classification results depend on the ability to always destabilize to a maximal tb example. Also it's also possible to destabilize to two distinct maximal tb representatives.

3. Relation to contact geometry

Given a Legendrian knot L in (M, ξ), there is a natural $tb - 1$ surgery called *Legendrian surgery.* Legendrian $tb - 1$ surgery on a knot in a fillable contact structure is fillable.

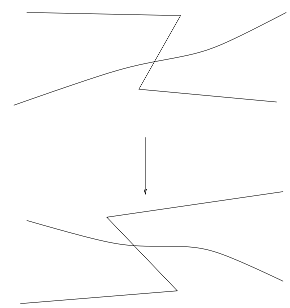

FIGURE 1. Basic move.

Question 2. *Does Legendrian surgery preserve tightness?*

The answer is no (due to Honda), if you allow manifolds with boundary. Does there exist a closed example?

Project: Distinguish contact structures using Legendrian knots.

Question 3. *Can you always distinguish tight contact structures ξ_1 and ξ_2 on the same manifold M by finding a Legendrian knot of a certain tb (or twisting number) and r which exists in one but not in the other?*

For example, consider the Lisca-Matić examples. These are tight structures on $\Sigma(p,q,pqn-1)$, obtained by Legendrian surgery on distinct Legendrian knots. The contact structures are distinguished by the Chern classes of Stein 4-manifolds which bound $\Sigma(p,q,pqn-1)$. Compare Seiberg-Witten-Floer basic classes with homotopy classes of 2-plane fields which have fillable (or tight) contact structures.

Understand relationship with Ozsvath-Szabo invariants.

The tight contact manifolds $(T^3 = \mathbf{R}^3/\mathbf{Z}^3, \xi_n)$, $n \in \mathbf{Z}^+$, given by the 1-forms $\alpha_n = \sin(2\pi nz)$ $dx+$ $\cos(2\pi nz)dy$, are distinguished by the maximal twisting number among curves isotopic to $x = y = const.$

Question 4. *Consider two Legendrian knots L_1, L_2 which are not isotopic but have the same tb and r. Let (M,ξ_1) and (M,ξ_2) be two contact manifolds obtained by performing Legendrian surgery on L_1 and L_2, respectively. (For example, we take L_1 and L_2 be the maximal tb examples of 5_2.) Then are ξ_1 and ξ_2 isotopic?*

Question 5. *Let L be a maximal tb Legendrian knot in (S^3,ξ_{std}). Is $(S^3 \setminus L, \xi_{std}|_{S^3\setminus L})$ universally tight? (Honda conjecture: Yes.) Maybe also true if L has nonmaximal tb but is also nonstabilized. More realistically, what about just $\mathbf{Z}$-covers of $S^3 \setminus L$? (Etnyre conjecture: Honda conjecture is false but $\mathbf{Z}$-cover statement is true.)*

4. Relation to topology

Relate Legendrian knot invariants to the Alexander module. (The conjectures above are relevant to this.) Find a relationship of Legendrian knot invariants with the Alexander norm or Thurston norm.

According to Fuchs-Tabachnikov, there are no finite-type invariants of Legendrian knots besides tb, r, and standard finite-type invariants of topological knot. Can you extract nonfinite type topological knot invariants by looking at the set of Legendrian knots of a given oriented topological knot type? For example, is the maximal tb invariant a nonfinite type invariant? One of the advantages apparent in Legendrian knot theory is that Legendrian knots are very sensitive to orientation reversal (very chiral).

Consider the smooth knot concordance group $\mathcal{C}$. $\mathcal{C}$ is a direct sum of countably many $\mathbf{Z}$'s and $\mathbf{Z}/2\mathbf{Z}$'s and maybe $\mathbf{Z}/4\mathbf{Z}$ factors. Are there any $\mathbf{Z}/4\mathbf{Z}$ factors? (Try to show there are none.) If there are $\mathbf{Z}/4\mathbf{Z}$ factors, are there countably many such factors? (The $\mathbf{Z}/2\mathbf{Z}$ factors come from the fact that amphichiral knots have order 2.)

Sliceness: Let $\overline{tb}(K)$ be the maximal tb invariant for a given oriented topological knot type K. If $\overline{tb}(K) \geq 0$, then K is not slice and K is of infinite order in $\mathcal{C}$. Can we improve Livingston, Kirk, etc.'s results (e.g., find simpler proofs) on some knots in Rolfsen's knot table not being slice.

Conjecture 1 (Etnyre conjecture)**.** *Either a knot is of order 2 or concordant to a knot with $\overline{tb} \geq 0$ or its mirror with max $\overline{tb} \geq 0$.*

Look up theorems of Gompf and Cochran on knot concordance (using gauge theory). Try to one-up them using contact geometry.

Legendrian knot complement problem is easily true (using Eliashberg's classification of tight contact structures on (S^3, ξ_{std}). However, the topological knot complement problem requires the ingenuity of Gordon and Luecke. Try to use the Legendrian version to reprove Gordon-Luecke.

5. Contact homology

Problem 4. *Are there useful contact homology invariants for stabilizations?*

Problem 5. *Combinatorially understand the full symplectic field theory for Legendrian curves.*

Problem 6. *Let L be a Legendrian knot (link) inside (S, ξ_{std}). If (M, ξ) is the Stein fillable contact structure obtained by performing Legendrian surgery on L, then is there a way to relate the contact homology of (M, ξ) from the differental graded algebra (DGA) associated to L? Also try to do this for arbitrary Legendrian links L in a tight manifold.*

Stabilization questions: If (M, ξ) in the above problem is obtained from a stabilized Legendrian knot, then is the contact homology of (M, ξ) simple? (Is is almost trivial? Does it equal the base field?)

6. Relationship to Riemannian geometry

Problem 7. *Try to introduce meaningful Riemannian geometry into contact topology. Is there a preferred metric for a contact structure? Can you relate to sectional curvature? Do you need to use coarse geometry?*