(TAO LI)

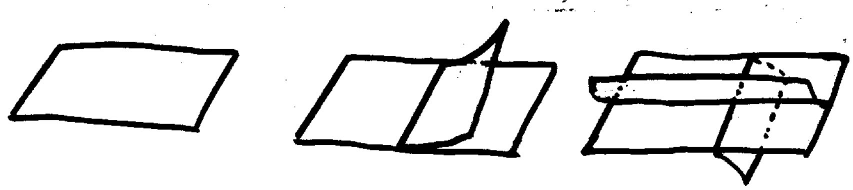
Let M be a closed irred. ori. 3-mfld.

Def. <u>Essential laminations</u> in M

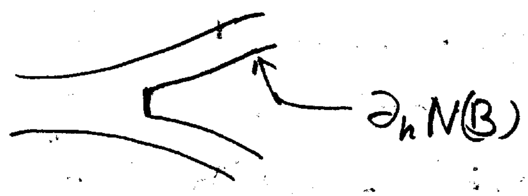
1. Every leaf is π_1-inj.
2. No sphere leaf.
3. End-incomp: 

Ex.
- Reebless foliations
- 2-sided incomp. surfaces
- $C \times$ a surface

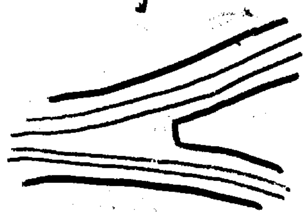
Def. <u>Branched Surfaces</u>



$N(B)$, $\partial_v N(B)$



A lamination is carried by a branched surface B



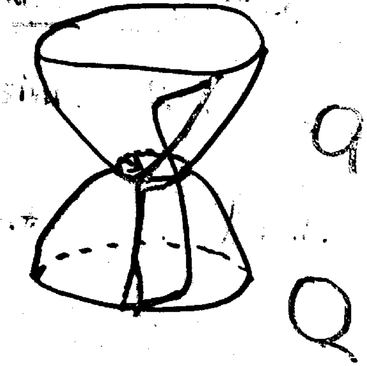
Relation between essential laminations and branched surfaces?

Thm (Gabai - Oertel)

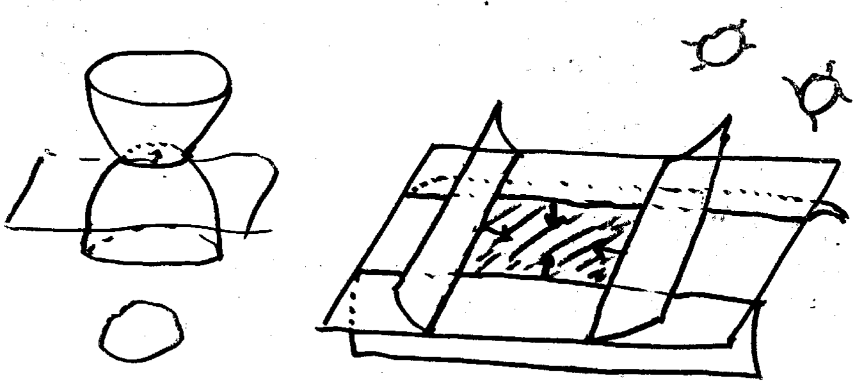
- Every essential lamination is carried by a branched surface with the following properties:
  1. $\partial_h N(B)$ is incomp.
  2. No monogon
  3. No Reeb component
  4. No disk of Contact
- If a lamination is carried by a brached surface as above, then it is an essential lamination.

Thm (Gabai - Oertel) S^3 contains a branched surface with the 4 properties.

Twisted disk of contact



Q: What properties make a branched surface "useful"?



Def. <u>sink disk</u>: the branch direction of every ∂ arc points inwards.

Def. <u>Laminar branched surface</u>

1 - 3 + No sink disk

Thm.1. Suppose M contains a laminar branched surface B

- $\widehat{M} \cong \mathbb{R}^3$
- If $M - \mathring{N}(B)$ is not a union of I-bundles and M is ator., then $\pi_1(M)$ is word hyp.

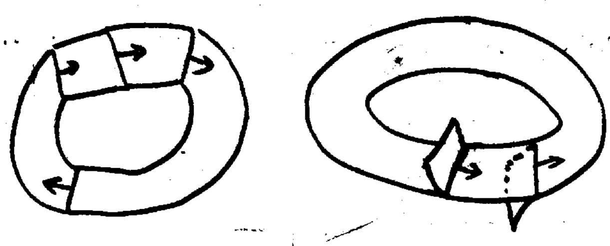
Thm (a) Every l. b. s. fully carries an essential lamination.

(b) Any ess. lamination that is not a lamination by planes is fully carried by a l. b. s.

(c) If λ is a lamination by planes, any branched surface that carries λ must contain a sink disk.

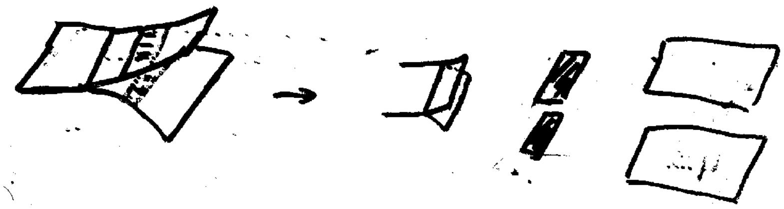
Remark: If M contains a lamination by planes, $M \cong S^1 \times S^1 \times S^1$

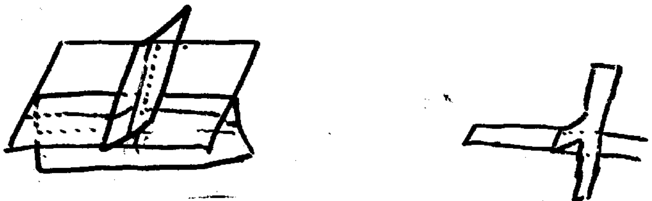
part (c):



Proof of part (a):

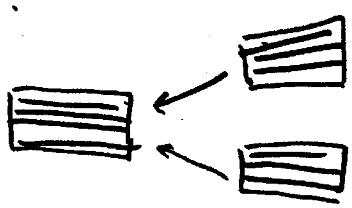
- decompose B into pieces.



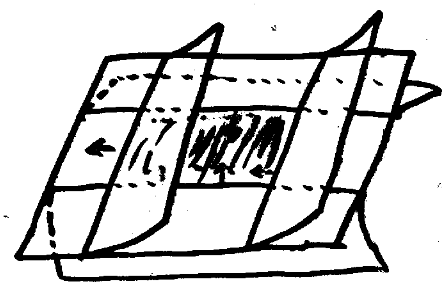


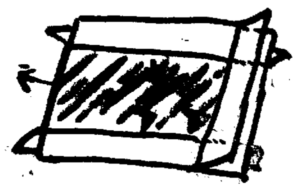
P(L)

- Construct a lamination carried by P(L).



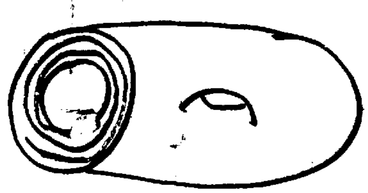
• Extend this lamination through the rest of B.





$B - \underline{\underline{P(L)}}$

1. non-planar branches

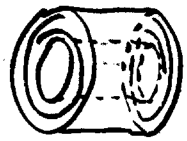


$\because \ \forall f : I \to I \ \begin{pmatrix} f(0)=0 \\ f(1)=1 \end{pmatrix}$

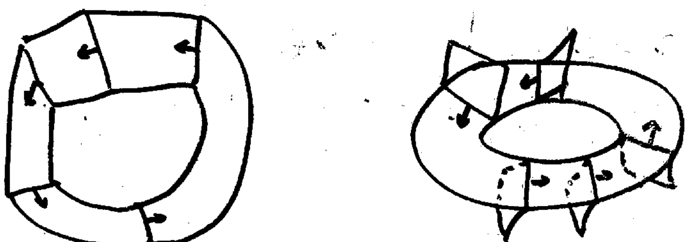
is a commutator

i.e. $\exists g, h,\ f = g \circ h \circ g^{-1} \circ h^{-1}$

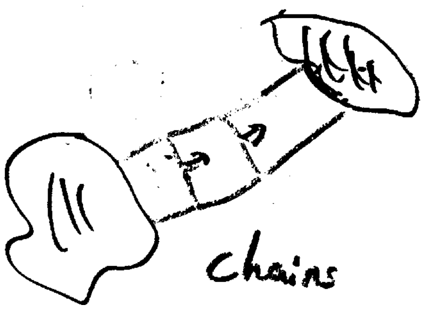
2. annuli



3. disks



cycles



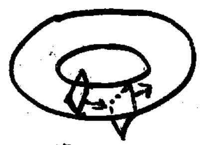
chains

Proof of part (b).

λ is an ess. lamination

- Take enough many essential simple closed curves.

- Construct "good" branched surfaces near these curves.

 or 

- split λ toward these "good" parts.

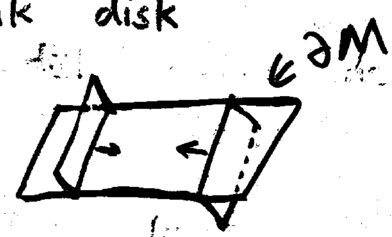
Applications:

Consider branched surface with ∂.

$$(B, \partial B) \hookrightarrow (M, \partial M)$$

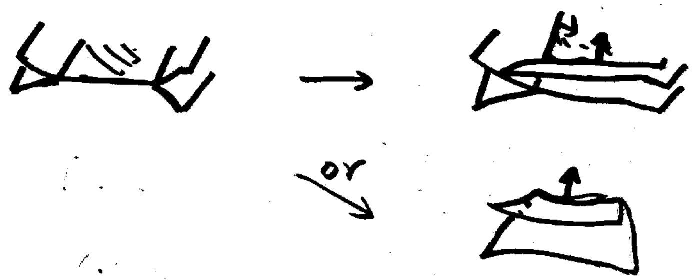
Suppose $\partial M = T^2$

Half sink disk



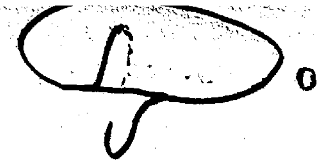
Suppose B has no sink disk and no half sink disk.

Split $\mathcal{B}$ near ∂B, s.t. ∂B becomes a union of circles.



No half sink disk $\Rightarrow$ never see a sink disk during the splitting

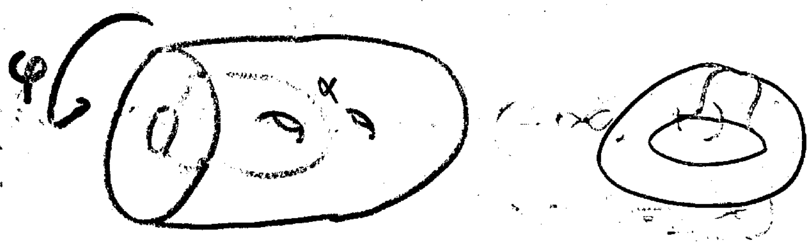
Now, do Dehn filling to M and cap off ∂B.

Ex. If $\partial B =$ 

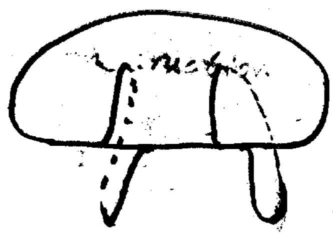
get any slope in $(-1, 0)$

Thms of Roberts

fibered knots

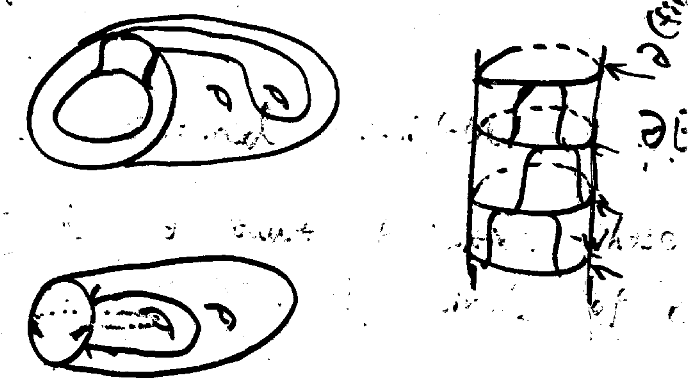


If α and $\varphi(\alpha)$ are disjoint, $\exists$ a l. b. s.



slopes in $(-1, \infty)$

If $\alpha \cap \varphi(\alpha) \neq \emptyset$ put in some



If k is a hyp. fibered knot.

the worst case $\longrightarrow (-\infty, 1)$ $[-1, 1]$

or $(-1, \infty)$

Remark: If a similar construction can be done using 2 product disks, can get more slopes and prove property P for fibered knots.

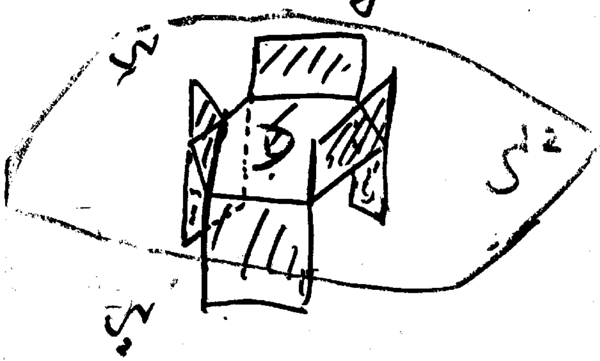
- Non-fibered knots.

Gabai's sutured manifold decomp.

Thm (Gabai) $\exists$ taut foliation whose ∂ is a foliation by circles of c slope.

$\Rightarrow$ For any knot in S^3, taut foliations can realize any slope in an interval $(-a, b)$ $a, b >$

- Murasugi Sum



D: $2n$ gon

$n=1$: connected sum

$n=2$: plumbing

Let k be a knot, and S be a min. genus Seifert surface.

$$S = S_1 \#_M S_2$$

$$k = L_1 \#_M L_2 \qquad L_i = \partial S_i$$

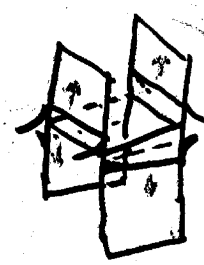
D can be viewed as a monkey saddle.

Ex.

Thm (Gabai) S has min. genus iff S_i has min. genus for both i.

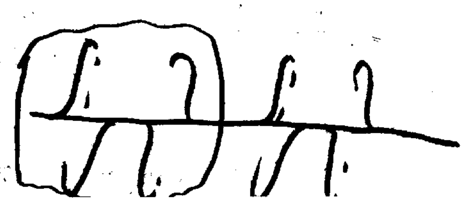
$S^2 - D$ is a decomposing disk.

branched surface:



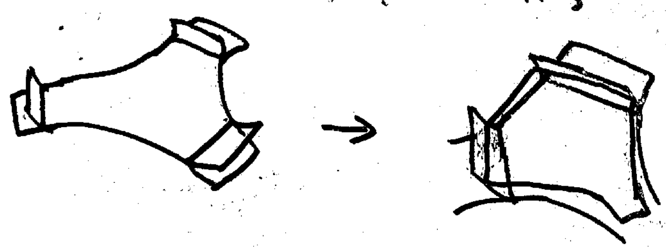
∂ train track:

$(-n, n)$



- If $S_i - D$ is not a disk (for both i), then $\exists$ taut foliations realizing slopes in $(-n, n)$ ($\Rightarrow$ Property P)

If $S_1 - D$ is a disk, then



can get rid of sink disks

- $2n$- Murasugi sum.
$\exists$ taut foliations realizing slopes in $(-(n-1), n-1)$.

If $n \neq 2$, get property P.

Plumbing case:

suppose $S_1 - D$ is a disk, then S_1 is an annulus.

- knotted annulus $\Rightarrow$ k is a satellite knot ✓.
- If both S_1 and S_2 are unknotted annuli $\Rightarrow$ k is a 2-bridge knot ✓

So:

In this case, $\exists$ another decomposing disk that yield slopes in $(-\infty, 1)$ or $(-1, \infty)$