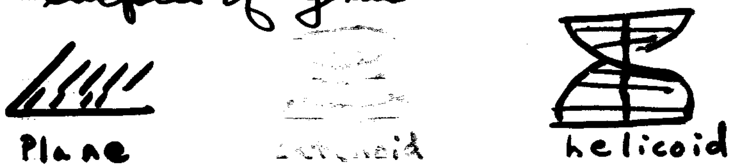
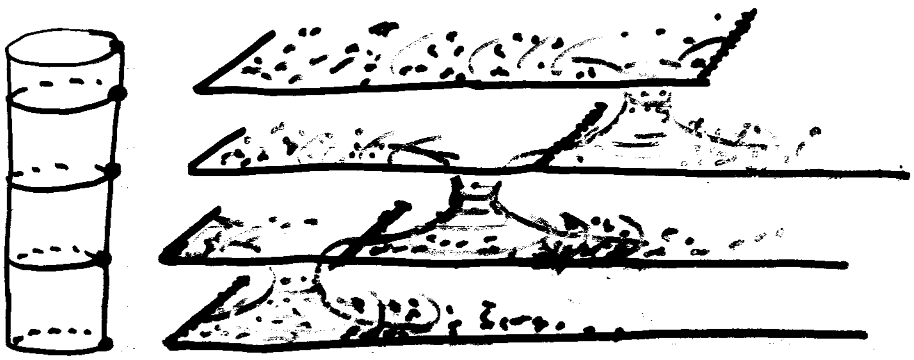
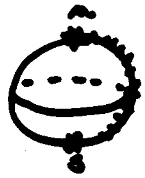
Examples of properly embedded minimal surfaces of genus = 0.







<u>Conj</u> (Meeks) If $E \subset M \subset \mathbb{R}^3$ isp an end of genus 0, then E converges exponentially to one of the above examples.

<u>Conj</u> (Meeks) If $M(i) \subset M^3, \langle\,\rangle$ is a seq of embedded minimal surfaces with locally bounded genus, then a subsequence converges to a minimal lamination $\mathcal{L}$ with smooth leaves and S = finite set of

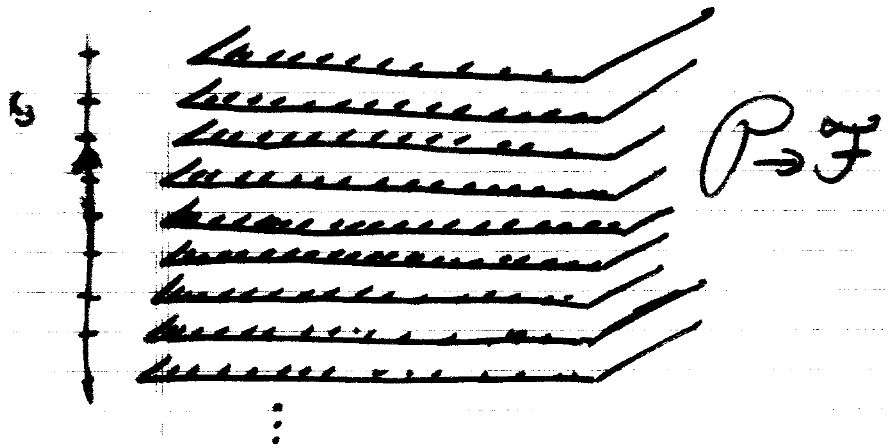
Homothetic Blow Down

Def A foliation $\mathcal{F}$ of $\mathbb{R}^3$ is a homothetic blow down of $M^2 \subset \mathbb{R}^3$ if $\exists\ t_i \in \mathbb{R}^+$, $t_i \to 0$ such that

$$t_i M = \{(t_i x_1, t_i x_2, t_i x_3) \mid x \in M\}$$

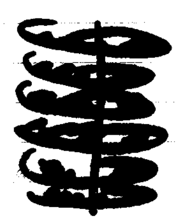
converges to $\mathcal{F}$ as $i \to \infty$.

Example Let P = set of planes at height n, $n \in \mathbb{Z}$. Then $\frac{1}{n} P \to \mathcal{F}$ = foliation by horizontal planes.



Ex 2 M = helicoid (vertical)

Then $\frac{1}{n}M \to \mathcal{F}$ = foliation by horizontal planes. The singular set S of non-smooth convergence is the x_3-axis.



Blow down of R_t.

$\frac{1}{n} R_t \to \mathcal{F}$-foliation by horizontal planes.

S = straight red line.

Conj (Meeks) Suppose
$E \subset M$ is a representative
of an end of M with genus
0. Then E converges asymptotically
to the end of a classical
example.

Conjecture (Meeks) If $M(i) \subset M^3$, <>
is a sequence of compact embedded
minimal surfaces with locally
bounded genus, then a subsequence
converges to a minimal lamination
of M^3. The singular set of
convergence is a finite set of
points and C^2 curves.

Thm If $M \subset \mathbb{R}^3$ is a p.m.s with more than one end, then M has a unique limit tangent plane at ∞ which passes thru the origin.

Thm Suppose $M \subset \mathbb{R}^3$ is a properly embedded min surface. Then $\mathcal{E}(M)$ has an natural ordering $L : \mathcal{E}(M) \to [0,1]$.

Def A highest end in the ordering is called the top end and the lowest in the ordering the bottom end. The other ends are called middle ends.

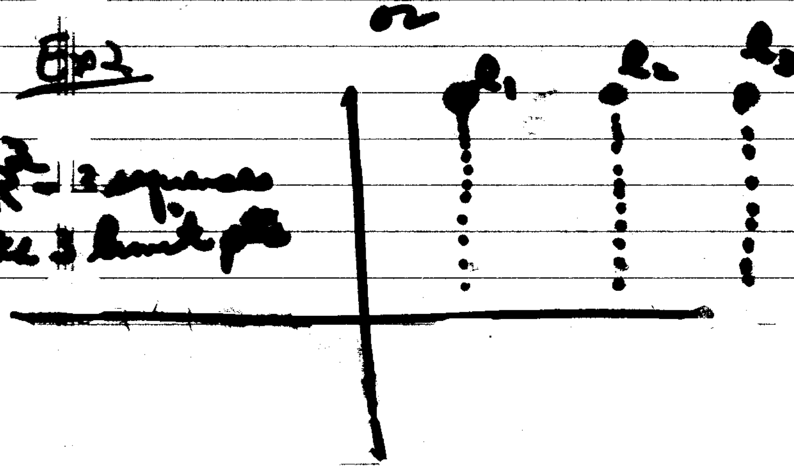
Thm The middle ends of $M \subset \mathbb{R}^3$ have quadratic area growth. In particular, M can have at most 2 limit ends and so M has a countable number of ends.

Ex.1 $\mathbb{R}^2 - \text{Cantor} \not\subset \mathbb{R}^3$

or

Ex.2

$\mathbb{R}^2 - 3$ sequences with 3 limit pts

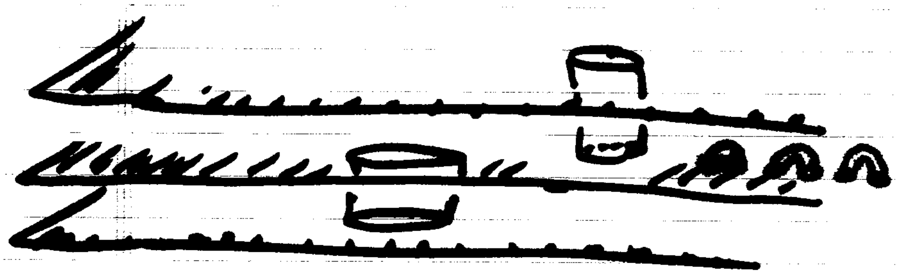


Thm Topological Classification Thm

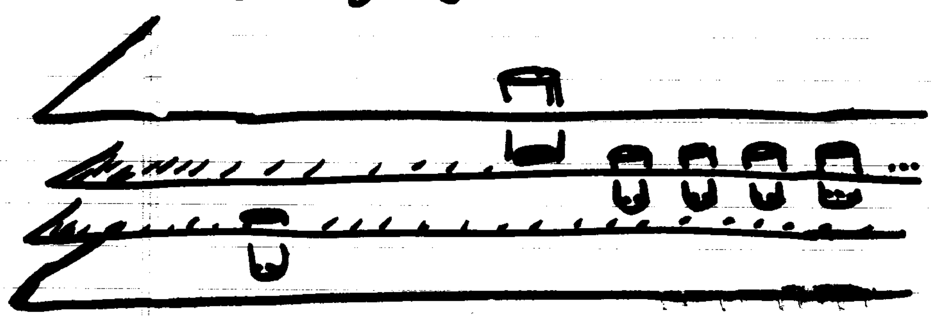
Suppose $M_1, M_2 \subset \mathbb{R}^3$ are 2 properly embedded minimal surfaces. Then M_1, M_2 are properly isotopic $\Longleftrightarrow$ $\exists$ a homeo $h : M_1 \to M_2$ preserving the ordering of the ends and the odd-even property of middle ends.

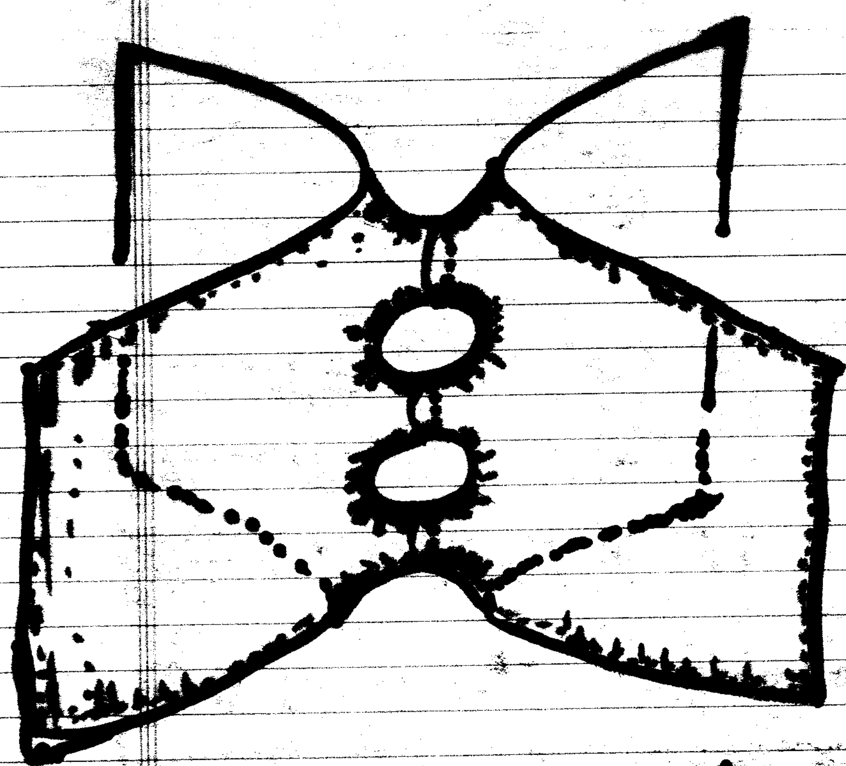
Note that if M_1, M_2 have the same genus and 1 end then they are isotopic.

ODD-type end

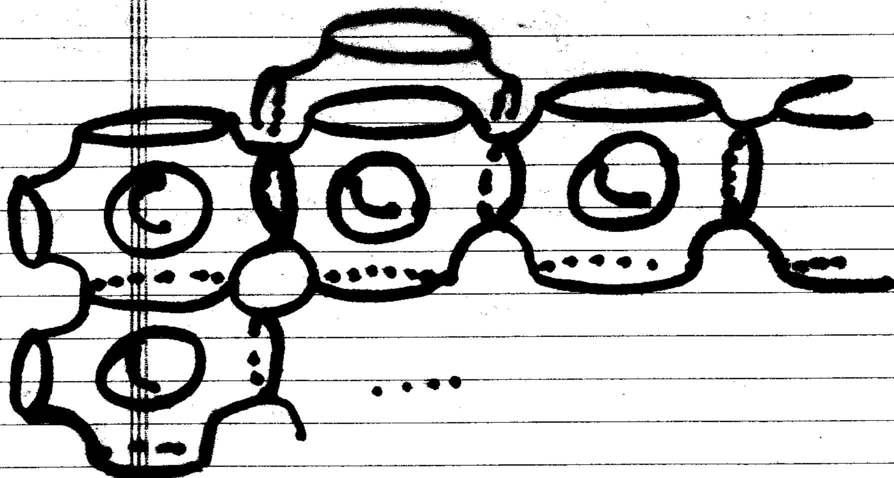


Even-type end
(always g=∞)





Sherk 1-periodic
minimal surface
1 end g = ∞.



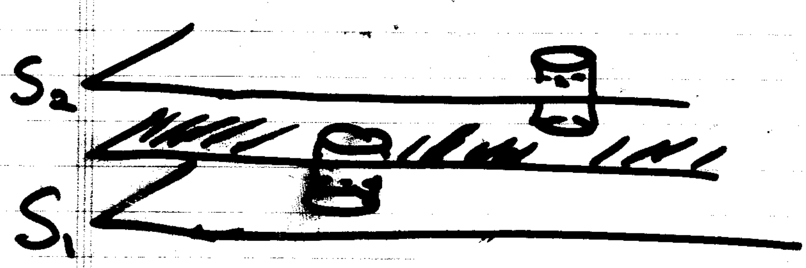
Schwartz Primitive
Surface 3-periodic
$g = \infty$ 1 end.

<u>Thm</u> A doubly-periodic minimal surface has infinite genus and 1 end.

Proof of Classification

Step 1 Construct a proper family $\mathcal{P} = \{ P_\alpha \mid \alpha \in W \subset \mathbb{Z} \}$ of properly embedded standard parallel planes such that $P_\alpha \cap M = 1$ curve. Haken moves.

Step 2 Prove that M intersect slab between P_α and $P_{\alpha+1}$ is standard



Conj (Meeks) If $M_1, M_2 \subset H^3$ are 2 homeomorphic properly embedded constant mean curvature 1 surfaces, then they are properly ambiently isotopic.

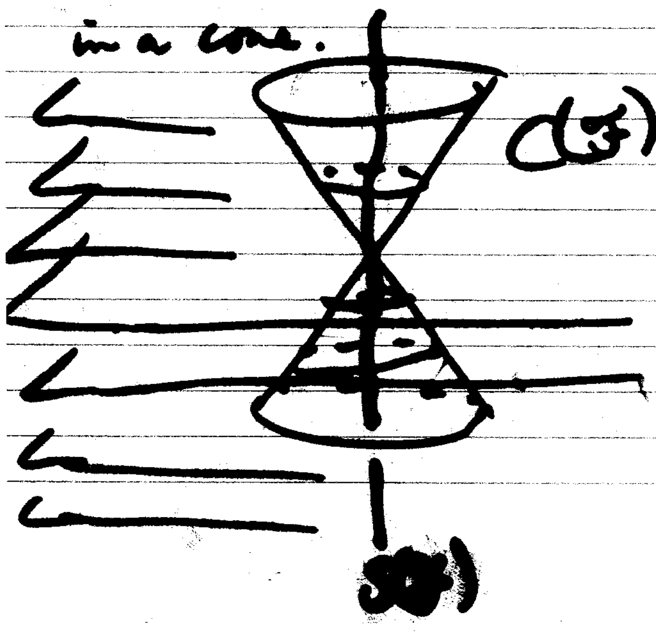
Conj (Meeks) The middle ends of a proper minimal $M \subset \mathbb{R}^3$ are C^0-asymptotic to a plane or a catenoid.

Theorem (Meeks - Rosenberg)

If $M \subset \mathbb{R}^3$ is a properly embedded minimal surface, then each annular end of M is asymptotic to the end of a plane catenoid or helicoid. Furthermore if M has genus 0 and finite topology then M is a plane, catenoid or helicoid.

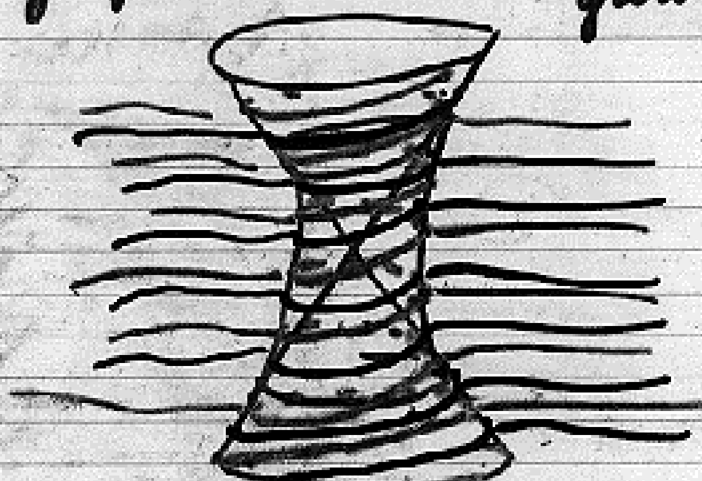
Step 1 (Colding-Minicozzi)

Suppose $M \subset \mathbb{R}^3$, proper minimal, $\pi_1(M) = 0$, $M \neq$ plane. Then there is foliation of $\mathbb{R}^3$ $\mathcal{F}$ by parallel planes such that given $t_i \to 0$, $t_i M \to \mathcal{F}$. with $S(\mathcal{F})$ = Lipschitz curve in a cone.

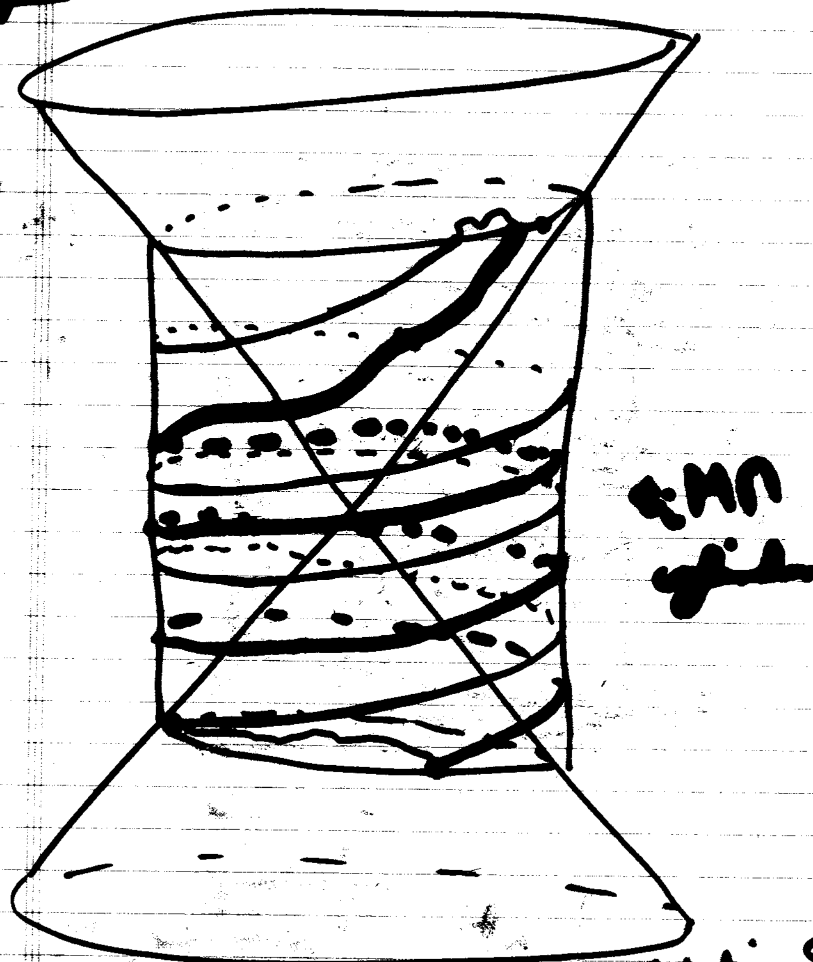


Properties 1. $S(\mathcal{F})$ intersects each plane in $\mathcal{F}$ in a single point

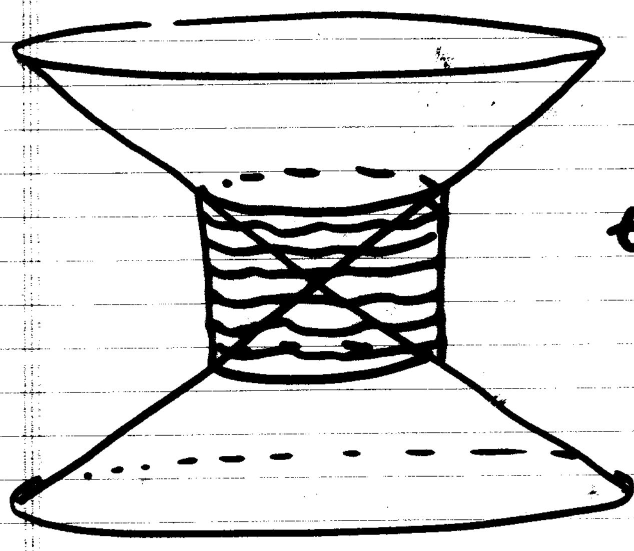
2. Outside the cone $C(\mathcal{F}) \cup B(r)$, r large, M is a 2 sheeted multi graph over $\mathbb{R}^2$ with sublinear growth.



Step 2 $\mathcal{F} \pitchfork M$



Find C^1-perturbation of S^1-foliation, S_ε of cylinder st. each S^1 intersects blue and orange curves in 1 point.



$t_i M$

S_t foliation $-1 \le t \le 1$

Fill in by minimal disk graphs to get foliation of solid cylinder by minimal disks $\{D_t\} = \mathcal{D}_i$

[redacted]

By Morse Theory,

$\mathcal{D}_i \pitchfork t_i M$ and so

$\frac{1}{t_i}\mathcal{D}_i \pitchfork M$. But

$\frac{1}{t_i}\mathcal{D}_i \to \mathcal{F}$ and since

the Gauss map of M is open

$\mathcal{F} \pitchfork M$.

Assume $\mathcal{F}$ is horizontal,

then $g : M \to S^2 = \mathbb{C} \cup \{\infty\}$

misses $\{0, \infty\}$ and so

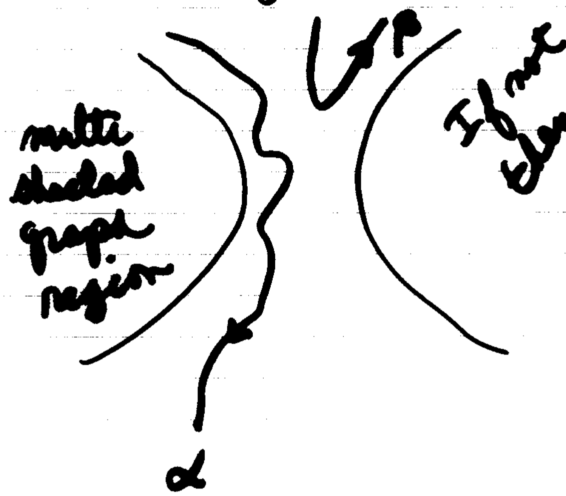
$g(p) = e^{f(p)}$; $e^z : \mathbb{C} \to \mathbb{C} - \{0\}$

Step 3 Each plane in $\mathcal{F}$ intersects M in a single arc. This implies $M = \mathbb{C}$ and $x_3 = \mathrm{Re}(z)$.

Need to prove that if not, then there is a proper domain $G \subset M$ that is a graph over an infinitely disconnected subdomain in $P \in \mathcal{F}$ with boundary values in P.

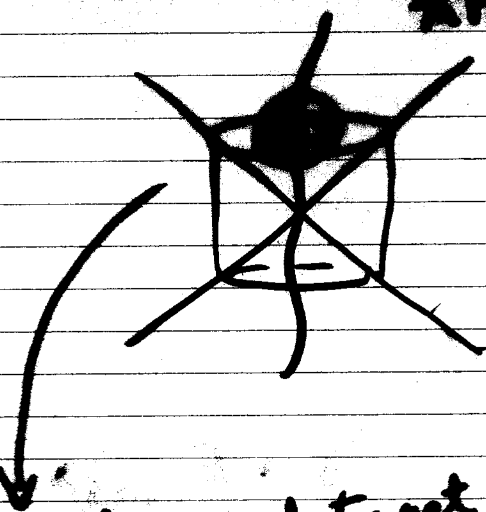
Step 4 If M has bounded Gaussian curvature, then $g(z) = e^{az+b}$ and (if M is embedded) $a \in i\mathbb{R}$ and $b = 0$. Hence M = helicoid.

Step 5 Reduce to bounded curvature case. Need to show $g(z) = e^{h(z)} \Rightarrow h(z) = az+b$.

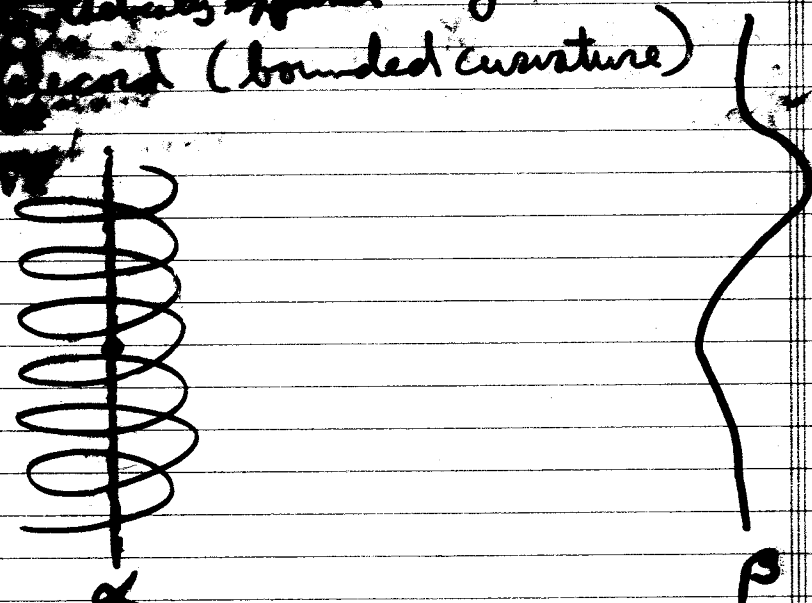


If not shown g^{-1}(equator) = 2 or more arcs α, β.

[illegible]



[illegible] locally expand to get vertical [illegible] (bounded curvature)



Cor If $M \subset \mathbb{R}^3$ is a properly embedded minimal surface with one end and finite genus then 1. $M \underset{conf}{\approx} \bar{M} - \{p_\infty\}$

2. $dx_3 + i\,dx_3^*$ and $\frac{dg}{g}$ are meromorphic 1-forms on $\bar{M}$ (after a rotation of M).

In particular the space of minimal surfaces $M_{g,e}$ with genus g and e ends is an analytic variety.

Conj (Meeks) If G is a minimal graph over a proper subdomain $D \subset \{x_3 = 0\}$ with zero boundary values, then G has at most 2 components not contained in $\{x_3 = 0\}$.

Conj If $M \subset \mathbb{R}^3$ has finite genus, then M has bounded Gaussian curvature.

Conj If $M \subset \mathbb{R}^3$ is a proper minimal surface with infinite topology and $g = 0$, then M is a Riemann example R_t.

Conformal Structures

<u>Thm</u> A properly embedded minimal surface M in $\mathbb{R}^3$ with 2 limit ends is recurrent for Brownian motion.

<u>Proof</u> Show $\exists$ a plane that intersects M in a compact set. Use that $\ln(r^2+1) - x_3^2$ is superharmonic on a minimal surface.

<u>Conj</u> (Meeks) If M is a properly embedded minimal surface in $\mathbb{R}^3$ with more than 1 end then M is recurrent for Brownian motion. If M is a properly embedded minimal surface, then + harmonic fns on M are constant.

Pitts-Rubinstein Conj

If M^3, $\langle\,\rangle$ is a bumpy metric, then there is a bound on the index of embedded minimal surfaces in M^3 with fixed genus.

$\Longrightarrow$ If $G \subset \mathrm{Diff}(S^3)$ is a finite subgroup, then $G = f G' f^{-1}$ for some $G' \subset O(4)$.

Thm (Meeks-Scott)

M^3 has a geometric structure, not spherical or hyperbolic, and $G \subset \mathrm{Diff}(M)$ is finite, then G is conjugate to a subgroup $G' \subset \mathrm{Isom}(M^3)$.

OBS (Meeks-Yau, Thurston)

A finite subgroup of $\mathrm{Diff}(\mathbb{R}^3)$ is conjugate to a subgroup of $O(3)$