(1)

Topology of algebraic hypersurfaces

$$V^n \subset \mathbb{C}P^{n+1}$$

non-singular algebraic hypersurface

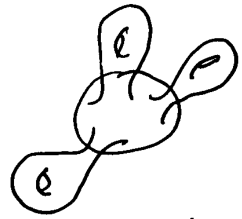
V is the zero set of a polynomial f of deg d

As a complex variety V depends on the coefficients of f.

But as a smooth manifold and even as a symplectic manifold V is completely determined by n and d.

Given n and d what can we say about V ?

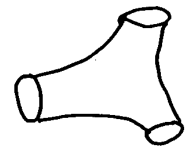
<u>$n=1$ case</u>:



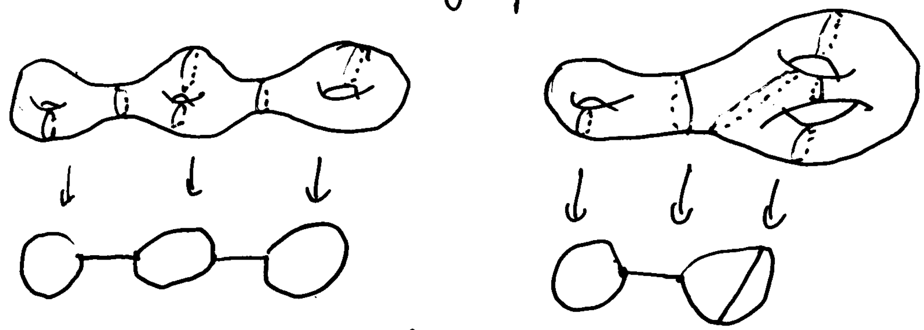
V is a Riemann surface

sphere with g handles, $g = \frac{(d-1)(d-2)}{2}$

one can decompose any such surface into pairs of pants

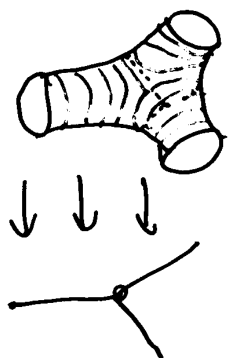


Such a decomposition may be encoded by a 3-valent graph



More than encoding:

These 3-valent graphs serve as bases of singular fibration:



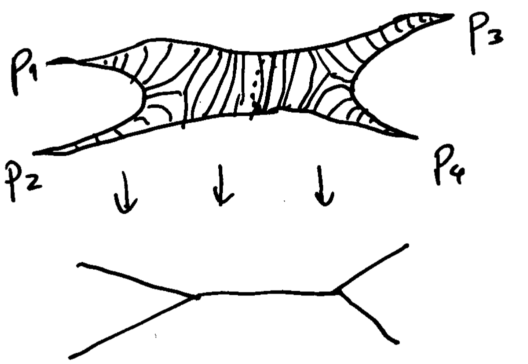
singular fiber  over the 3-valent vertex

These fibrations glue to a fibration on V

(3) $n=1$ "Boundary" case

$(V; p_1, \dots, p_k)$ is a Riemann surface with marked points

Its decomposition to pairs of pants may be expressed as a fibration over a 1-and 3-valent graph



If $V \subset \mathbb{P}^2$ is a smooth curve we have a natural choice for marked points

$(\mathbb{C}^*)^2 = \mathbb{C}\mathbb{P}^2 \setminus 3$ coordinate axes

p_i are intersection with the axes

The goal: Find an analogue of pairs of pants decomposition for projective (and some other) hypersurfaces (and complete intersections) in higher dimensions

(4)

$n = 2$ case: 4-manifolds

$d=1$	$d=2$	$d=3$	rational surfaces
$\mathbb{CP}^2$	$\mathbb{CP}^1 \times \mathbb{CP}^1$	$\mathbb{CP}^2 \# 6\overline{\mathbb{CP}}^2$	$K<0$
			$p_g = 0$

$\left(p_g = h^{2,0} \text{ geometric genus} \right)$

$d=4$ $(K=0)$ $p_g = 1$ K3-surface

$$\text{K3} \to S^2$$

one way to present it: singular toric fibration

 S^2 with 24 "cone" points

half dimensional (as in $n=1$ case)

$d=5$?? V_5 is not decomposable to # or $\times$

how to describe its topology?

homotopy type: $p_g = 4$ $\chi = 55$ $\sigma = -35$,

But what is it as a smooth manifold?
(and symplectic)

$n > 2$?? or $\overset{n=2}{d > 5}$?

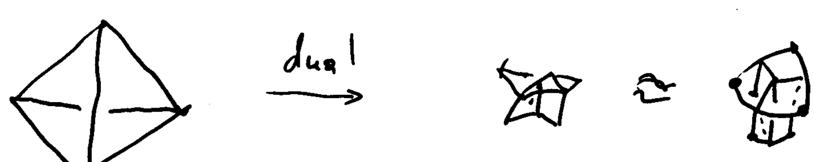
(5) Higher-dimensional counterparts of 3-valent graphs:

<u>def</u>: A generic polyhedron Π of dim n is the n-skeleton of the cellular decomposition dual to a (possibly singular) triangulation of an $(n+1)$-dimensional manifold M.

Similarly we may define a generic polyhedron with boundary or even with corners by letting M to be an $(n+1)$-manifold with boundary or with corners (e.g. M is a convex polytope)

<u>E.g.</u>: $n=1$ Π is a 3-valent graph or 3- and 1-valent graph

$n=2$ Π is a "special spine"



Note that Π has a natural cellular decomposition

⑥ Statement of the main theorem

Let $V \subset \mathbb{CP}^{n+1}$ be a non-singular hypersurface of degree d. Recall that n, d determine its smooth and symplectic type

Thm (DIFF)

$\exists\ \Pi$ - a generic polyhedron with corners

and $\lambda : V \to \Pi$ a smooth map which is a <u>stratified fibration</u>, i.e. the restriction of λ to each open cell of Π is a trivial fibration and, furthermore, that the smooth type of degeneration of the fiber over smaller-dimensional cells depends only on the type of the cell (see example below)

<u>such that</u>

- $\Pi \overset{\text{hom. equiv}}{\simeq} \bigvee_{p_g} S^n$

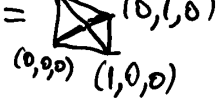
  the wedge of $p_g = h^{n,0}$ copies

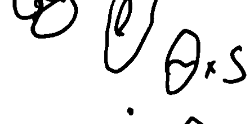
- $H^n(\Pi) \xrightarrow{\lambda^*} H^n(V)$ is injective

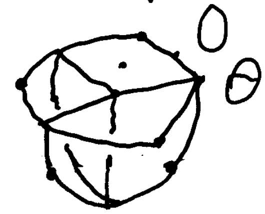
Π is obtained from a lattice triangulation of the Newton polyhedron of V

<u>Example</u>: $n=2, d=1$

$f(x,y,z) = x+y+z+1$

$\Delta =$ 

$4D^2 \to$ 

$\Pi =$ 

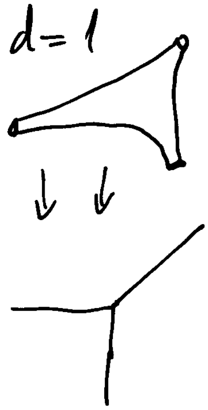
Continuation of the statement of the theorem: description of fibers.

- Over n-cells the fiber is T^n
- over interior $(n-1)$-cells the fiber is $\Theta \times T^{n-1}$
- Over interior $(n-2)$-cell the fiber is $\Big($  $\cup\, 4\, D^2\Big) \times T^{n-2}$ and so on...

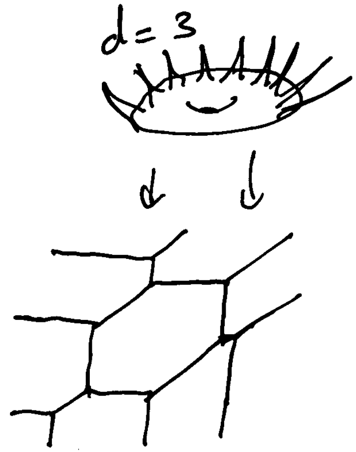
In general all fibers over interior cells are certain n-dim subcomplexes of T^{n+1}.
fibers over cells in the $(n+1)$-dimensional corners are the same as interior fibers for corresponding n-dim hypersurfaces

<u>Examples</u>: $n=1$

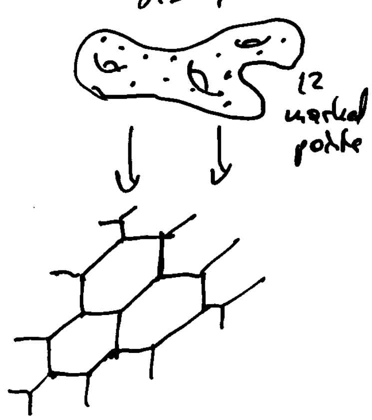
$d=1$



$d=3$

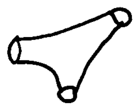


$d=4$



(7½)

Decomposition into pairs of pants in higher dimension.

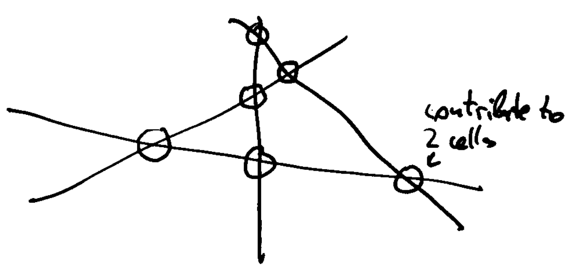
$P_1 =$  $\simeq \mathbb{CP}^1 \setminus 3$ pts

combinatorics of decomposition is described by a 3-valent graph

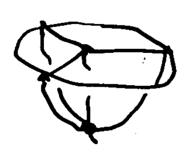
$P_n = \mathbb{CP}^n \setminus (n+2)$ hyperplanes in gen position

combinatorics is described by a generic polyhedron Π.

$n=2$ case



$\mathbb{P}^2 \setminus 4$ lines



(8) Symplectic structure of V

Thm (SYMP): There exist p_g embedded Lagrangian tori in V which are linearily independent in $H_n(V) \supset L_{tori}$

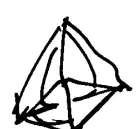
Furthermore, there exist p_g embedded Lagrangian spheres in V which are linearily independent in $H_n(V)$ and which span a subspace $L_{spheres} \pitchfork L_{tori}$

$$\begin{array}{c} L_{tori} \\ L_{spheres} \end{array} \left(\begin{array}{c|c} 0 & I \\ \hline \pm I & \cancel{\ast} \end{array} \right)$$

Thm' (symp): There exists a deformation ω_t, $0 \le t \le 1$ such that ω_t is symplectic for $t>0$, $\omega_1 = \omega$, $[\omega_t] = [\omega]$ $\forall t$ $\omega_0^n \geq 0$ such that λ is a Lagrangian fibration with respect to ω_0

$n=1$ case

$n=2$



all 4 cannot be Lagrangian

but where the fibers are smooth they are Lagrangian

⑨ Complex structure

Thm ($\mathbb{C}$): For any (n,d) we can choose $V \subset \mathbb{P}^{n+1}$ so that λ is a totally real
and λ
singular fibration

PROOF

Take $(\mathbb{C}^*)^{n+1} \subset \mathbb{C}\mathbb{P}^{n+1}$ $\qquad$ $\mathbb{C}^* \to \mathbb{R}$, $z \mapsto \log|z|$

torus part $V \cap (\mathbb{C}^*)^{n+1}$ →

$$\begin{tikzcd} V \arrow[d, "\mathrm{Log}"] & \subset & (\mathbb{C}^*)^{n+1} \arrow[d, "\mathrm{Log}"] \\ \mathcal{A} & \subset & \mathbb{R}^{n+1} \end{tikzcd}$$

$(z_1, \dots, z_{n+1}) \mapsto (\log|z_1|, \dots, \log|z_{n+1}|)$

amoeba ↗ (Gelfand-Kapranov-Zelevinski)

In general $\mathcal{A}$ may be very complicated but there is a deformation of V and $\mathcal{A}$ which straightens $\mathcal{A}$.

he same) Deformation is known as:

- Viro's patchworking (Viro* used it for construction of real algebraic varieties ~1982) or toric degeneration
- Maslov's dequantization (a part of a more general picture deforming part of $\mathbb{R}_+$ or the norm of $\mathbb{C}^*$ arithmetic operations

(10) Maslov's dequantization

deformation "classical" "quantum" $\mathbb{R}_+$
Idempotent semiring $\mathbb{R}_+^{\infty}$

deform addition in $\mathbb{R}_+ \ni x, y$

$$x \oplus_\alpha y = \sqrt[\alpha]{x^\alpha + y^\alpha} \qquad \alpha \geq 1$$

semiring ~~with respe~~ together with ×

<u>Claim</u>: For all finite $\alpha \geq 1$ $\mathbb{R}_+^{(\alpha)} \cong \mathbb{R}_+$ $(x \mapsto x^\alpha)$

$$\lim_{\alpha \to +\infty} (x \oplus_\alpha y) = \max\{x, y\}$$

Idempotent operation $x \oplus_\infty x = x$ – "dequantized" $\mathbb{R}_+$

Now instead of deforming V, i.e. deforming the coefficients of f we fix coefficients of f and deform arithmetic operations instead

(11)

Take a log point of view on this

$$\mathbb{R}_+ \xrightarrow{\log_t} \mathbb{R} \quad (\underbrace{\log(t)=\alpha}_{\text{before}}) \quad (\mathbb{C}^* \xrightarrow{\mathrm{Log}_t} \mathbb{R})$$

This equips $\mathbb{R}$ with a structure of semiring coming from $\mathbb{R}_+$: $x \oplus_t y = \log_t(t^x + t^y)$

$x \odot_t y = x + y$

$x \oplus_\infty y = \max\{x, y\}$

$x \odot_\infty y = x + y$

Take $\mathbb{R}^n$ and fix a polynomial, i.e. a collection of coefficients

$F_t : \mathbb{R}^{n+1} \to \mathbb{R}$ $\quad F_t(x,y) = a \oplus_t bx \oplus_t cy \oplus_t \dots$

it defines $\quad x = \log|z|,\ y = \log|w|$

$f_t : (\mathbb{C}^*)^{n+1} \to \mathbb{C}^*$ $\quad f_t(z,w) = t^a + t^b z + t^c w + \dots$

$$\begin{tikzcd} V_t = \{f_t = 0\} \arrow[d] & \subset & (\mathbb{C}^*)^{n+1} \arrow[d, "\mathrm{Log}_t"] \\ \mathcal{A}_t & \subset & \mathbb{R}^{n+1} \end{tikzcd}$$

<u>Lemma</u>: $\mathcal{A}_t$ converge in the Hausdorff metric to a certain non-Archimedean amoeba $\mathcal{A}$ (which is a PL n-dim subcomplex of $\mathbb{R}$)

12 Non-Archimedian amoebas:

A non-Archimedian field K is a field with valuation:

Def: (Minus-valuation) Real-valued

$v(x) = -\infty \iff x = 0$

$v(x+y) = \max\{v(x), v(y)\}$ $\qquad e^{v}$ is a norm

$v(xy) = v(x) + v(y)$

Example: K is Puiseux series in t

$a t^{1/3} + b t^{2} + \dots \qquad a, b, \dots \in \mathbb{C}$.

This is an algebraically closed field

A family of compex polynomials

$$f_t(z,w) = t^{a} + t^{b} z + t^{c} w + \dots$$

is a single polynomial over K

$$\begin{array}{ccc} V & \subset & (K^*)^{n+1} \\ \downarrow & & \downarrow \\ \mathcal{A}_K & \subset & \mathbb{R}^{n+1} \\ \uparrow & & \\ \text{Non-Archimedian} & & \\ \text{amoeba} & & \end{array}$$

(13)

Thm (Kapranov): A non-Archimedian amoeba $\mathcal{A}_K$ is completely determined by the extended Newton polyhedron of f_K

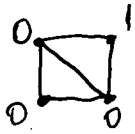
In $\Delta = \text{Convex hull}\{(j_1, \dots, j_{n+1}) \subset \mathbb{R}^{n+1} \mid a_{j_1, \dots, j_{n+1}} \neq 0\}$

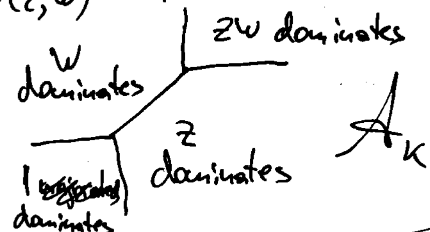
$$f = \sum_{j_1, \dots, j_{n+1}} a_{j_1 \dots j_{n+1}} z_1^{j_1} \cdots z_{n+1}^{j_{n+1}}$$

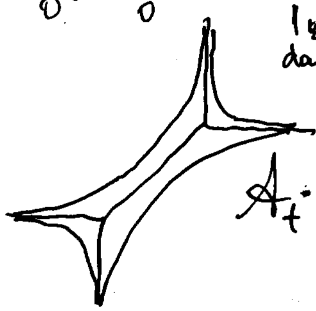
we fix the norms of the coefficients at each monomial.

Namely, $\mathcal{A}_K$ is the corner locus of the Legendrian transform of the valuation

Example: $f(z,w) = 1 + z + w + tzw$







$$x \oplus_t y \leq \max\{x, y\} + \log_t 2$$

$$\log_t(t^x + t^y) \leq \log_t(2\, t^{\max\{x,y\}})$$

(14) Relation to the real structure

Define $\gamma: V \to \mathbb{C}P^n$ - Logarithmic Gauss map

$V \subset (\mathbb{C}^*)^{n+1}$ - Lie group trivialized

Lemma: Critical locus of $\mathrm{Log}|_V : V \to \mathbb{R}^{n+1}$ coincides with $\gamma^{-1}(\mathbb{R}P^n)$

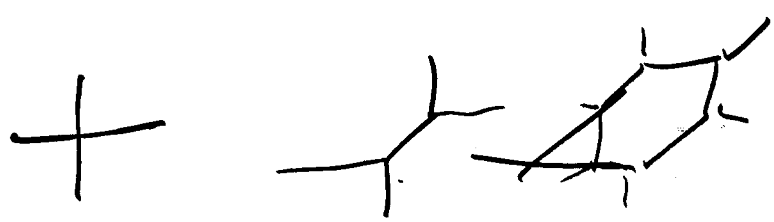
Thm: $\exists$ real structure on V such that

$$\mathbb{R}V = \gamma^{-1}(\mathbb{R}P^n)$$

(Maximal real structure)

It has p_g spherical components

Remark: $n=1$ These are so-called Simple Harnack curves (Harnack 1876)



15

Open questions

① Reverse Thm 1 (what do we equip Π with so that it determines V?)

② Can higher-dimensional pairs of pants help to describe complex structures on V?

③ Can λ be made into a "special Lagrangian" fibration in some sense?

e.g. $n=1$  is special w/respect to $z(dz)^2$, a quadratic differential

④ Relation to the Hodge structure:

Have filtration on ~~[illegible]~~ the cycles in V coming from cellular decomposition of Π

E.g. $n=2$ case the tori ~~[illegible]~~ ~~[illegible]~~ seem to be in $H^{2,0}_{lim}(V)=H^{0,2}_{lim}(V)$

i.e. $$\lambda^*(H^2(\Pi)) = H^{2,0}_{lim}(V) = H^{0,2}_{lim}(V)$$

How about $H^{p,q}$ $p,q>0$??