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There is no Lagrangian embedding of the Klein bottle into $\mathbb{C}^2$

- only open case in $\mathbb{C}^2$
- $\exists$ totally real embedding and Lagrangian immersion
- Hofer/Luttinger suggested approach using broken holomorphic curves (1996)

<u>Theorem:</u> Assume $K \subset \mathbb{C}P^2 \setminus H$ is a Lagrangian embedding of the Klein bottle. Then there exists an almost complex structure, standard near H, and a pseudoholomorphic sphere $G \subset \mathbb{C}P^2 \setminus K$, $[G] = [H]$, &

$$lk_{CP^2\setminus H}(G \setminus (G \cap H), \cdot) : H_1(K) \to \mathbb{Z}$$

is nontrivial.

· <u>Gromov</u>: $\exists$ symplectomorphism, s.t.

G & H are projective lines

$\Rightarrow$ · $K \subset \mathbb{C}^2 \setminus \mathbb{C}\times\{0\}$ with

$(\ell k_{\mathbb{C}^2}(\mathbb{C}\times\{0\}, .) : H_1(K) \to \mathbb{Z}) \neq 0$

· $\mathbb{C}^2 \setminus \mathbb{C}\times\{0\} \simeq \mathbb{C}\times(S^1\times(0,\infty))$

$\subset \mathbb{C}\times(S^1\times\mathbb{R})$ sympl.

· $\exists$ holomorphic disk $\Delta \subset \mathbb{C}\times(S^1\times\mathbb{R})$

$\partial\Delta \subset K$

$\Rightarrow$ · $d\theta = \omega$: $\int_{\partial\Delta}\theta > 0$, $\ell k(\partial\Delta, \mathbb{C}\times\{0\}) = 0$ ↯

Symplectic manifolds with convex and concave ends

- $K \subset \mathcal{U} \cong V \supset O_K \subset T^*K$: $\omega|_{\mathcal{U}} = d\theta|_V$.

  θ... Liouville form

- fix flat metric on K

  $\alpha := \theta|_{T_1^*K}$ contact 1-form

v.f. R_α : $R_\alpha \lrcorner\, d\alpha = 0$, $\alpha(R_\alpha) \equiv 1$

Reeb v.f., Reeb flow = geod. flow

- $(\mathcal{U}\setminus K, \omega) \simeq (T_1^*K \times (-\infty, t_0], d(e^t\alpha))$
- $(T^*K \setminus O_K, d\theta) \simeq (T_1^*K \times \mathbb{R}, d(e^t\alpha))$
- almost complex structures:

J on $\mathbb{C}P^2\setminus K$: $\omega(\cdot, J\cdot)$ Riemann

on the end $\begin{cases} J : \operatorname{Ker}\alpha \circlearrowleft \\ J(\frac{\partial}{\partial t}) = R_\alpha \end{cases}$

J_K on T^*K likewise: $J|_{\mathcal{U}\setminus K} = J_K|_{V\setminus O_K}$

- $J_\varepsilon(x) = \begin{cases} J(x) & \text{for } x \notin \mathcal{U} \text{ or } e^t > \varepsilon \\ J_K(x - \ln\varepsilon) & \text{else} \end{cases}$

- J_ε compatible to

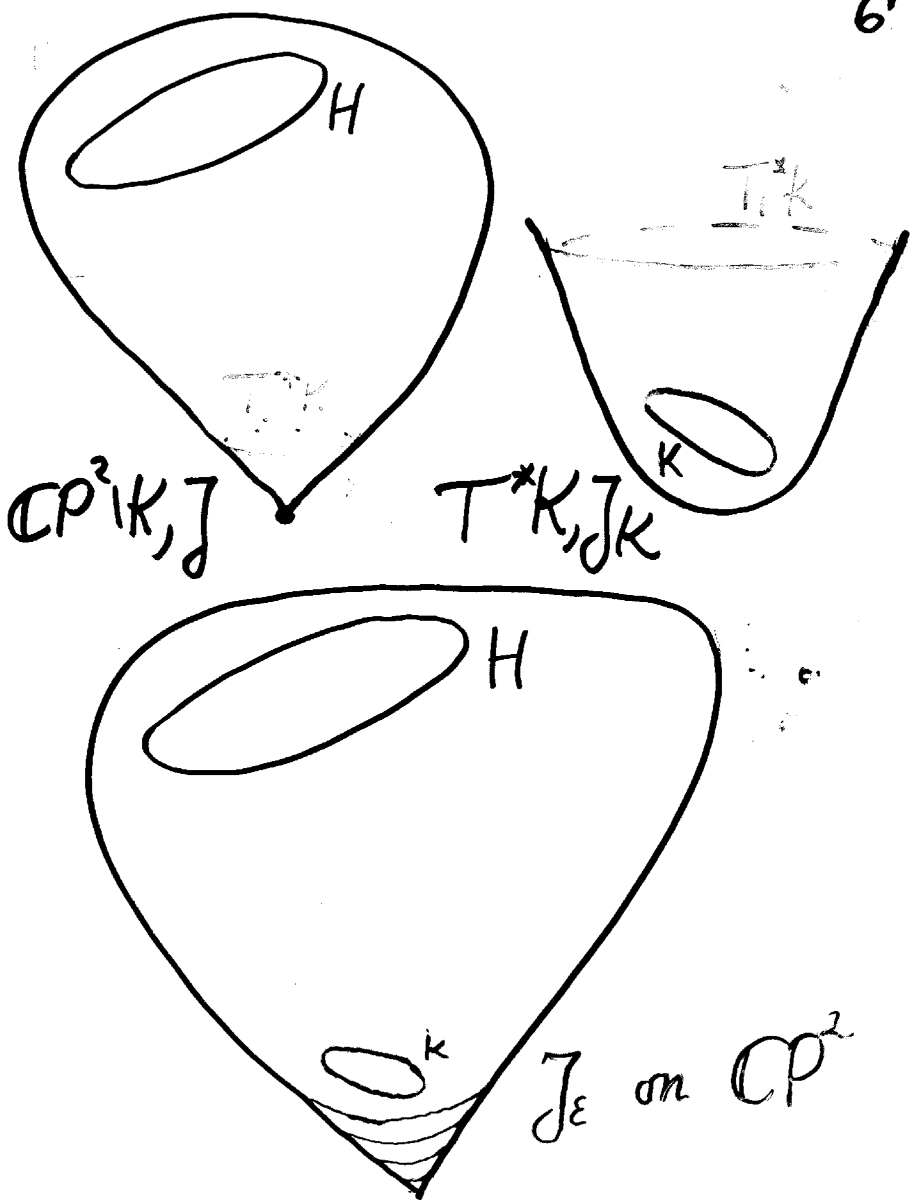
$$\omega_\varepsilon(x) = \begin{cases} \omega(x) & x \notin \mathcal{U} \text{ or } e^t > \varepsilon \\ \varepsilon \cdot (-\ln\varepsilon)^* d\theta & \text{else} \end{cases}$$

- $\omega_\varepsilon \simeq \omega$ isotopic, of course
- $J_\varepsilon \to J$ on $\mathbb{C}P^2 \setminus K$
- $J_\varepsilon(\,.\,+\ln\varepsilon) \to J_K$ on T^*K

% γ (closed) Reeb orbit

% $\Rightarrow$ $\gamma \times \mathbb{R} \subset T_1^*K \times \mathbb{R}$

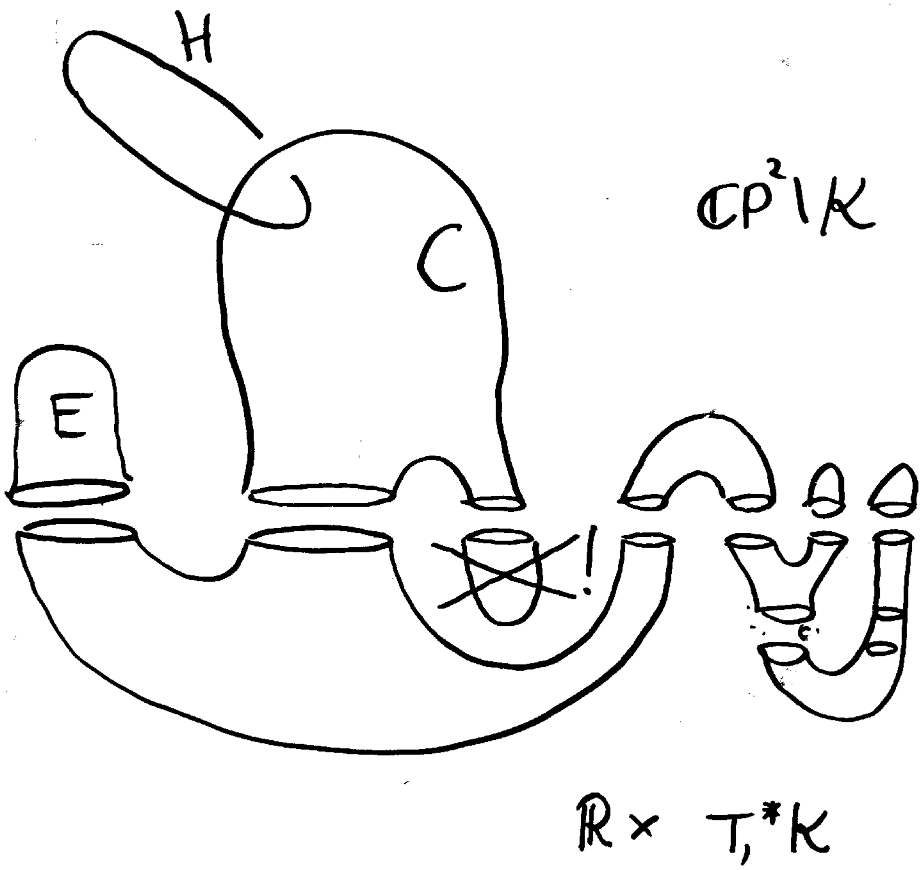
% J-holomorphic (cylinder)


H
$\mathbb{C}P^2 \setminus K, J$
T^*K, J_K
K
H
K
J_ε on $\mathbb{C}P^2$

Proof of Theorem

- Fix (generic) $p \in H$ & consider J-holomorphic deformations of H.
- $\exists$ non-compact sequence $\{u_n\}$ of such
- $u_{n'} \to u$ in the sense of Gromov:

- u "broken" J-holomorphic sphere
- consists of punctured spheres in $\mathbb{C}P^2 \setminus K$ & $T_1^*K \times \mathbb{R}$, asymptotic to cylinders over α-Reeb orbits
- $[u] = H$



schematic picture of u

- $\nexists$ contractible closed geodesic

$\Rightarrow$ $\nexists$ contractible closed α-Reeb orbit

$\Rightarrow$ $\nexists$ one-punctured J_K-holom. sphere in T^*K

$\Rightarrow$ $\cdot$ u contains a one-punctured J-holom. sphere E in $\mathbb{C}P^2 \setminus (K \cup H)$,

- E compactifies to smooth embed. symplectic disk $\overline{E} \subset \mathbb{C}P^2 \setminus H$ geodesic $\partial \overline{E} \subset K$

- **Strategy:** find unbroken J-holom. sphere G: $G \cdot E = 1$, $[G]=[H]$
- Fix $x \in E$ & $\xi \in T_x^J \mathbb{C}P^2$
- Gromov: $\exists!$ smooth J_ε-holom. sphere, v_ε, $[v_\varepsilon]=[H]$, through x, $T_x v_\varepsilon = \xi$.

$\Rightarrow$ • $v_{\varepsilon_n} \to v$ broken (J, J_K)-holom. sphere, $[v]=[H]$, through x, $T_x v = \xi$

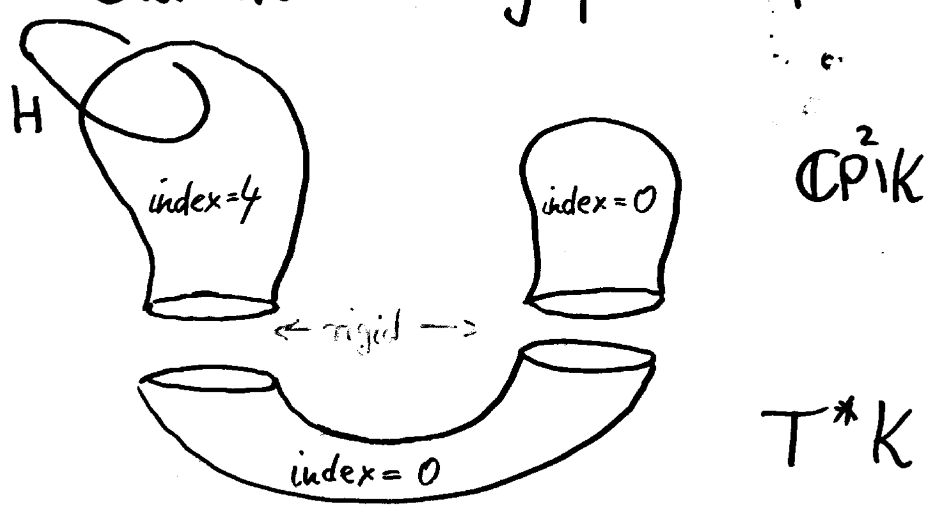
% all parts of u, v in $\mathbb{C}P^2 \setminus K$

% are simple and smooth

- (x,ξ) are 4 real parameters

$\Rightarrow$ · if index(compon. of v in $\mathbb{CP}^2 \setminus K$) < 4, then v must be unbroken for generic (x,ξ).

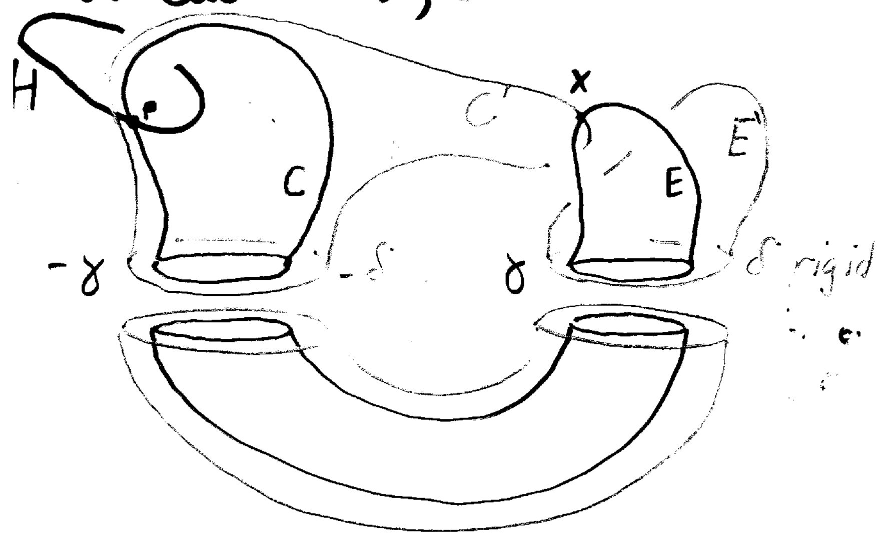
- But theoretically possible for v:



· index discussion, intersection arguments

& generic $p \in H \Rightarrow 3$ cases

1st case: u, v



· intersection $\Rightarrow \gamma \| \delta$

· Since $u_n \cdot C' \neq 0 \Rightarrow H \cdot C' \neq 0$

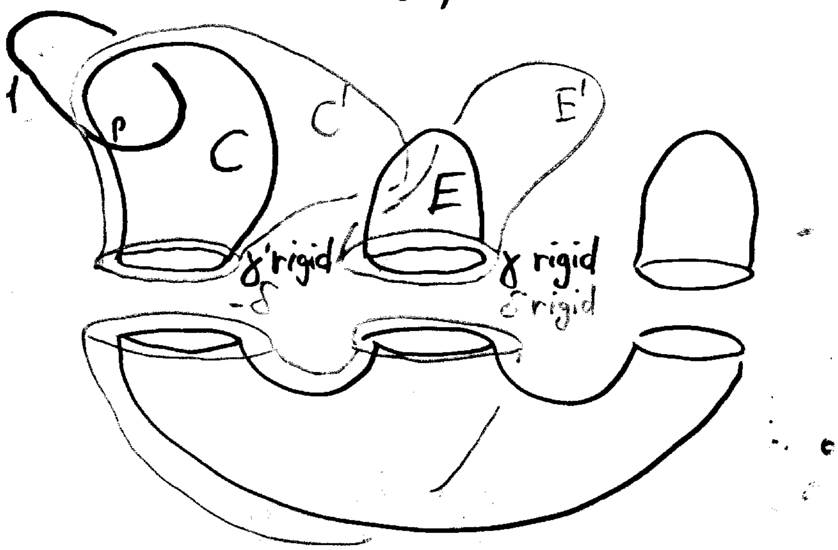
$$\cdot\ \underbrace{\int_\gamma \theta}_{} \overset{\int_E \omega}{\|} > 0, \quad \int_\delta \theta > 0 \qquad \int_{-\gamma(-\delta)} \theta < 0$$

$\Rightarrow \gamma, \delta$ on same Lagr. leaf in $T_1^* K$

$$(kC \# -\ell C') \cdot (kE \# -\ell E') = -k \ \lightning$$

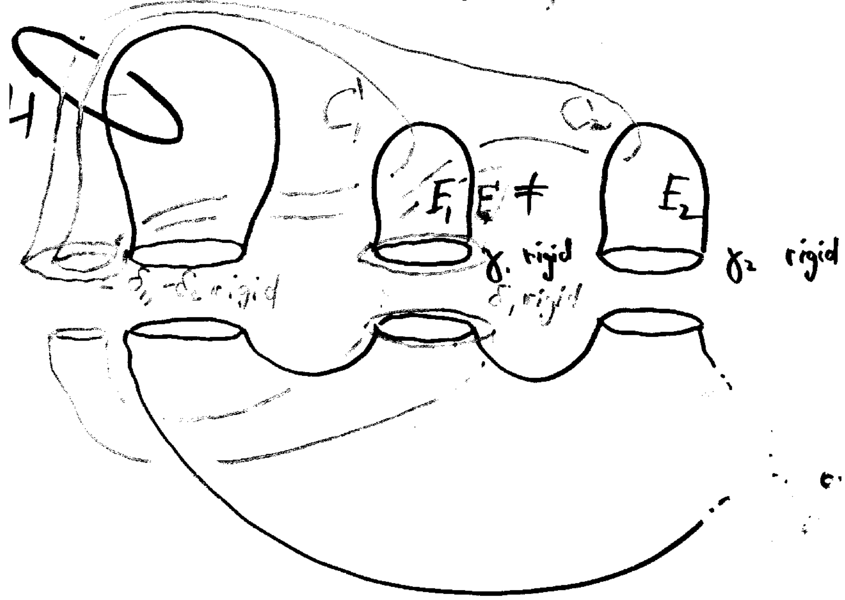
· Notice: all rigid geodesics are parallel

· 2nd case: u, v



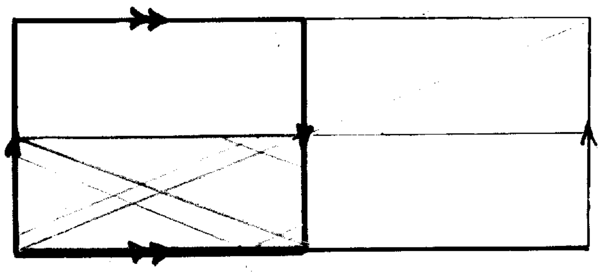
$$(2kC \infty - 2\ell C')\cdot(2kE \infty - 2\ell E') = -2k \;\lightning$$

· 3rd case: ν, ν_1, ν_2



$$(2kC_1' \multimap -2\ell C_2')\cdot(2pE_1 \multimap -2qE') > 4kp \;\lightning$$

Holomorphic curves in T^*K



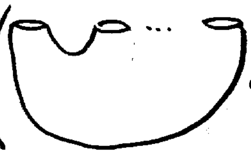
- only two simple rigid geodesics - odd
- all others come in 1-param. fam. - even
- Fix trivialization of $(T(T^*K), d\theta)$.

$\Rightarrow$ · Maslov class: $\mu : H_1(K) \to \mathbb{Z}$

$\Rightarrow$ · trivialization of $\xi = \ker(\theta_K|_{T_1^*K})$

$\Rightarrow$ • γ geodesic $\hat{=}$ closed α-Reeb orbit

$$CZ(\gamma) = \begin{cases} \mu(\gamma) & \gamma \text{ odd} \\ \mu(\gamma) + \frac{1}{2} & \gamma \text{ even} \end{cases}$$

$\Rightarrow$ • index( $\subset T^*K$) $= (-2 + 2e + o)$

e... # of punctures asympt. to even

o... # of pctrs asympt. to odd orbits

Proof: $\text{index} = (-2 + e + o) + \sum_i CZ(\gamma_i) + \sum_i \frac{1}{2} \dim_{\gamma_i}$

$$\sum_i [\gamma_i] = 0 \Rightarrow \sum_i \mu(\gamma_i) = 0 \quad \square$$

Broken holomorphic spheres in [H]

• index( $\subset \mathbb{C}P^2 \setminus K$)

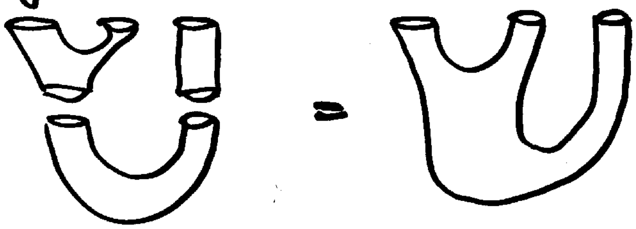
$$= (-2+o+e) - \sum_i \left(CZ(\gamma_i) + \tfrac{1}{2}\dim_{\gamma_i} \right) + 2c_1$$

$$= (-2+o+e) - \sum_i \mu(\gamma_i) + 2c_1$$

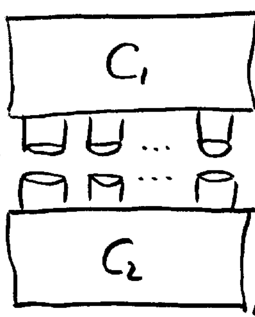
$$\equiv e \mod 2$$

since $\mu(\gamma) \equiv 1 \mod 2$ iff γ is odd.

• consider parts in T^*K & $T_1^*K \times \mathbb{R}$ together as broken curves in T^*K:



(only interested in its index)

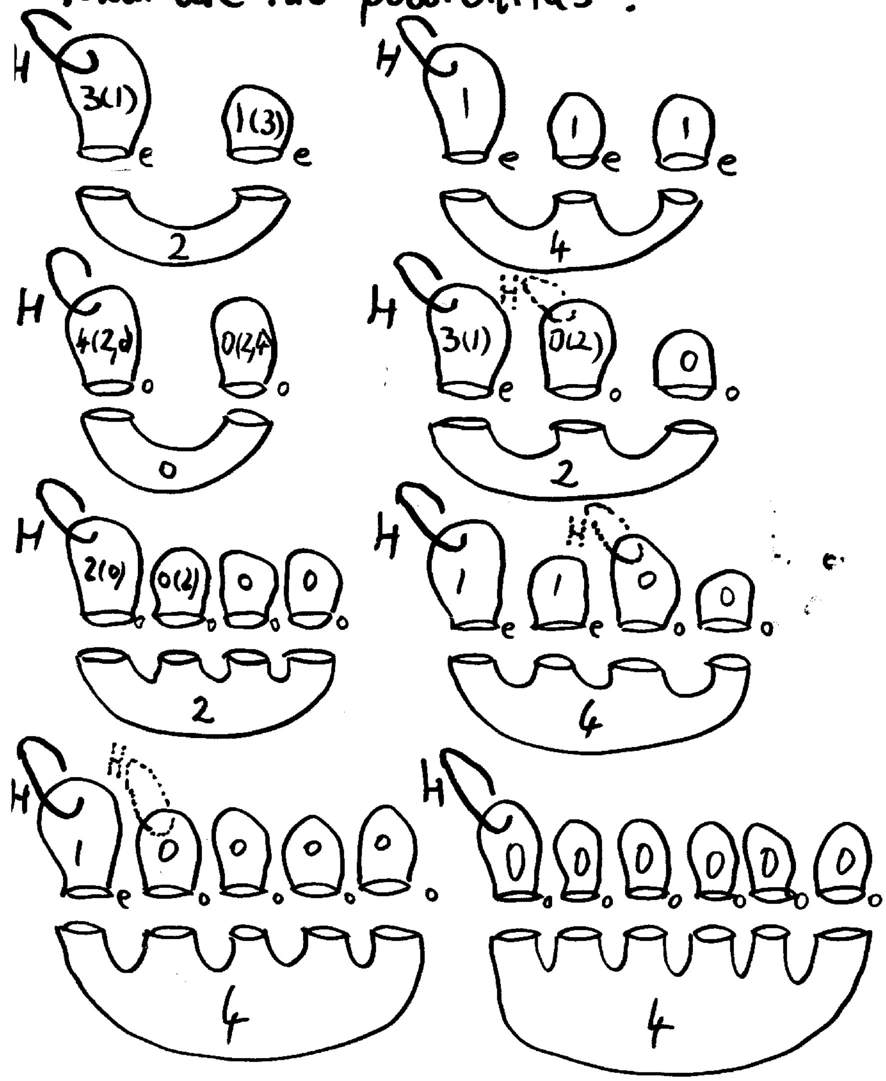
• index$\Big(C :=$ 

$\Big) = \text{index}(C_1) + \text{index}(C_2) - e$

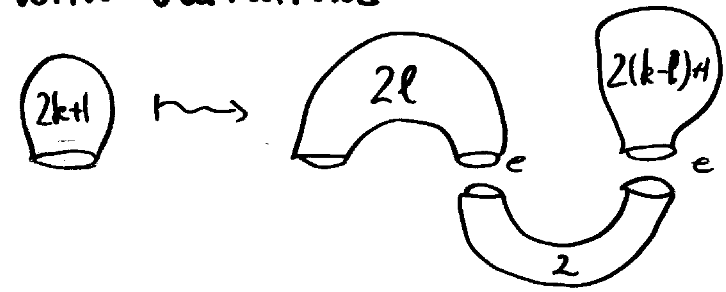
if $C = S^2$.

% "$-e$" due to coincidence relations

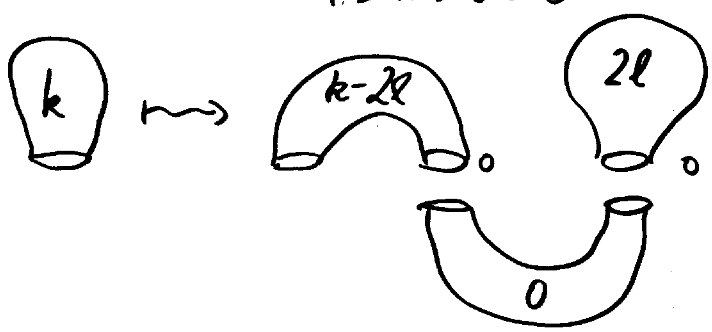
What are the possibilities?



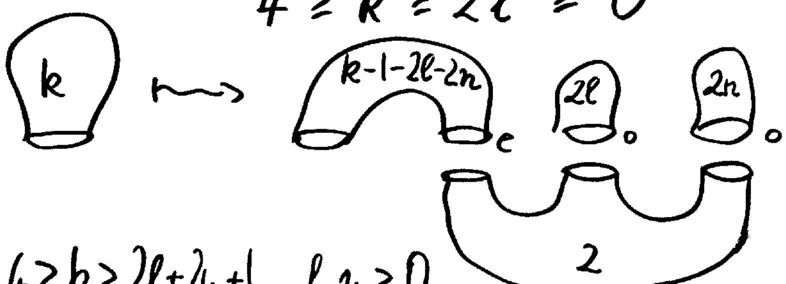
with variations



$1 \geq k \geq l \geq 0$



$4 \geq k \geq 2l \geq 0$



$4 \geq k \geq 2l+2n+1$, $l, n \geq 0$.

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