Invariants of Legendrian Knots and Links

Lenny Ng
M.I.T.

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Introductory definitions: M^3 compact.

Contact form on M: 1-form α such that

$$\alpha \wedge d\alpha \neq 0$$

everywhere on its domain.

Contact structure on M: a 2-plane distribution on M given locally by $\ker \alpha$, for some contact form α.

Contact manifold: a manifold with a contact structure.

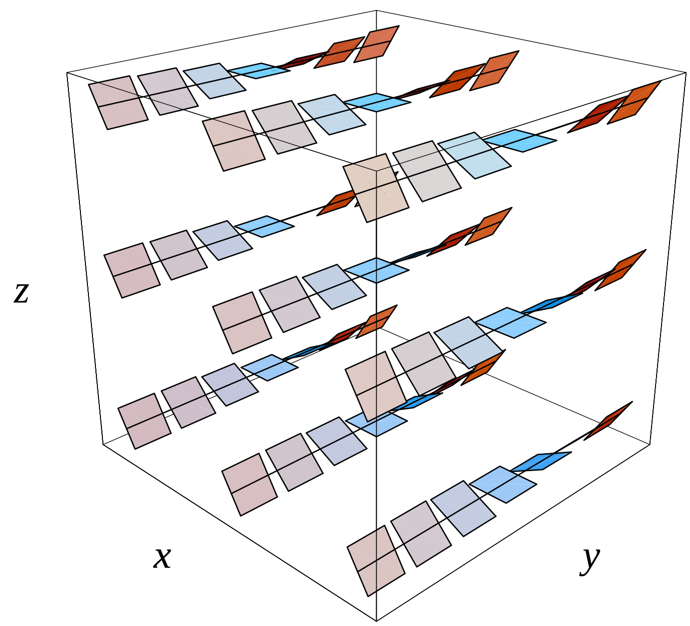
Legendrian (transversal) knot/link in a contact manifold: must be everywhere *tangent* (*transverse*) to the contact structure; i.e., α vanishes identically (never vanishes) along the knot/link.

Legendrian isotopic: two knots/links which have an isotopy of Legendrian knots/links between them.

Standard contact form on $\mathbb{R}^3$:

$$\alpha = dz - y\,dx.$$

Then $\alpha \wedge d\alpha = dx \wedge dy \wedge dz$.



The contact structure on $\mathbb{R}^3$ given by $\ker \alpha$.

By Darboux's Theorem, any contact manifold locally looks like standard contact $\mathbb{R}^3$.

Main question: How do we determine when two Legendrian knots/links in standard contact $\mathbb{R}^3$ are Legendrian isotopic?

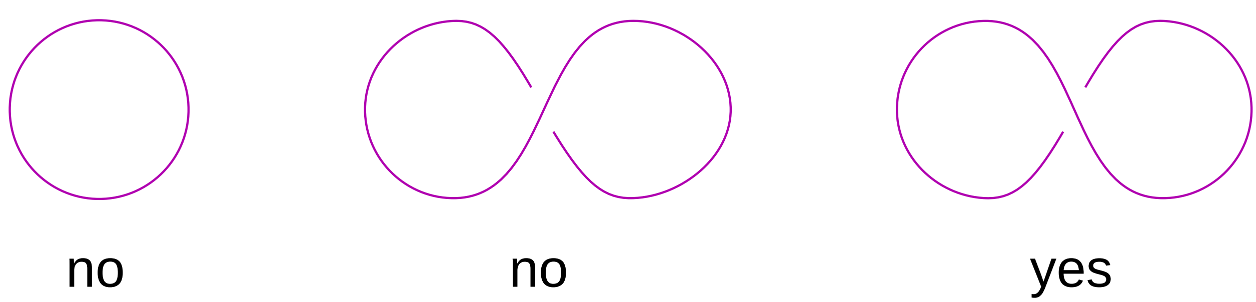
Two ways to depict Legendrian knots in standard contact $\mathbb{R}^3$.

Lagrangian projection: project to $\mathbb{R}^2_{xy}$. Recover third coordinate by setting $z = \int y\, dx$.

Front projection: project to $\mathbb{R}^2_{xz}$. Recover third coordinate by setting $y = dz/dx$.

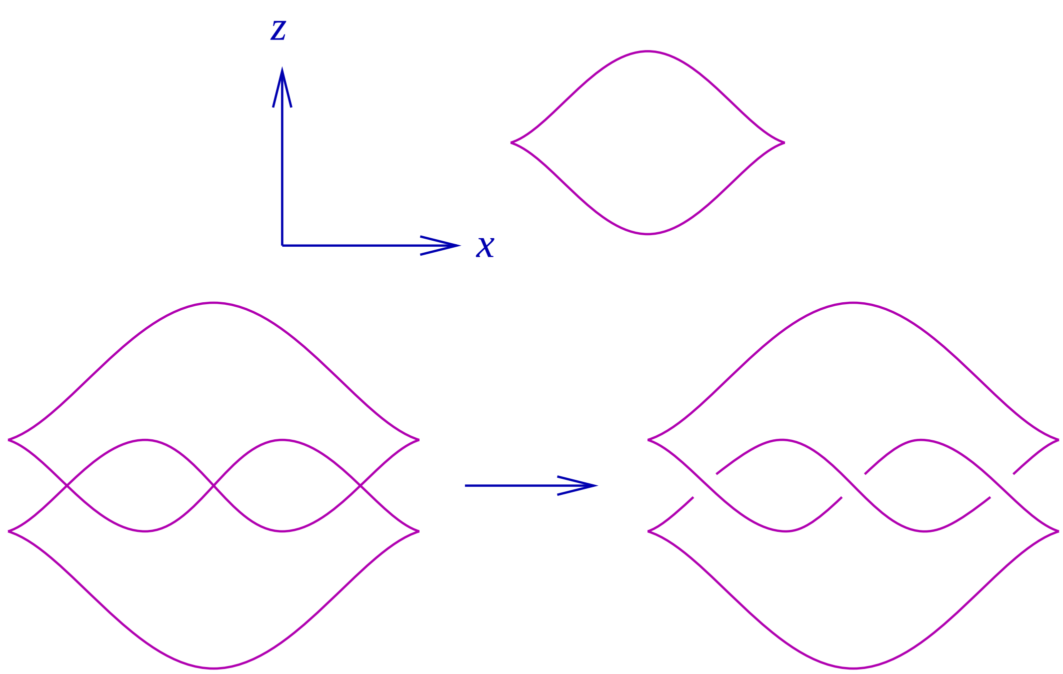
Problem with Lagrangian (xy) projection: not easy in general to tell when a knot diagram is the Lagrangian projection of a Legendrian knot.

- Signed area $\oint y\, dx$ must be zero.
- Inequalities on areas in knot diagram, resulting from over/undercrossing information.

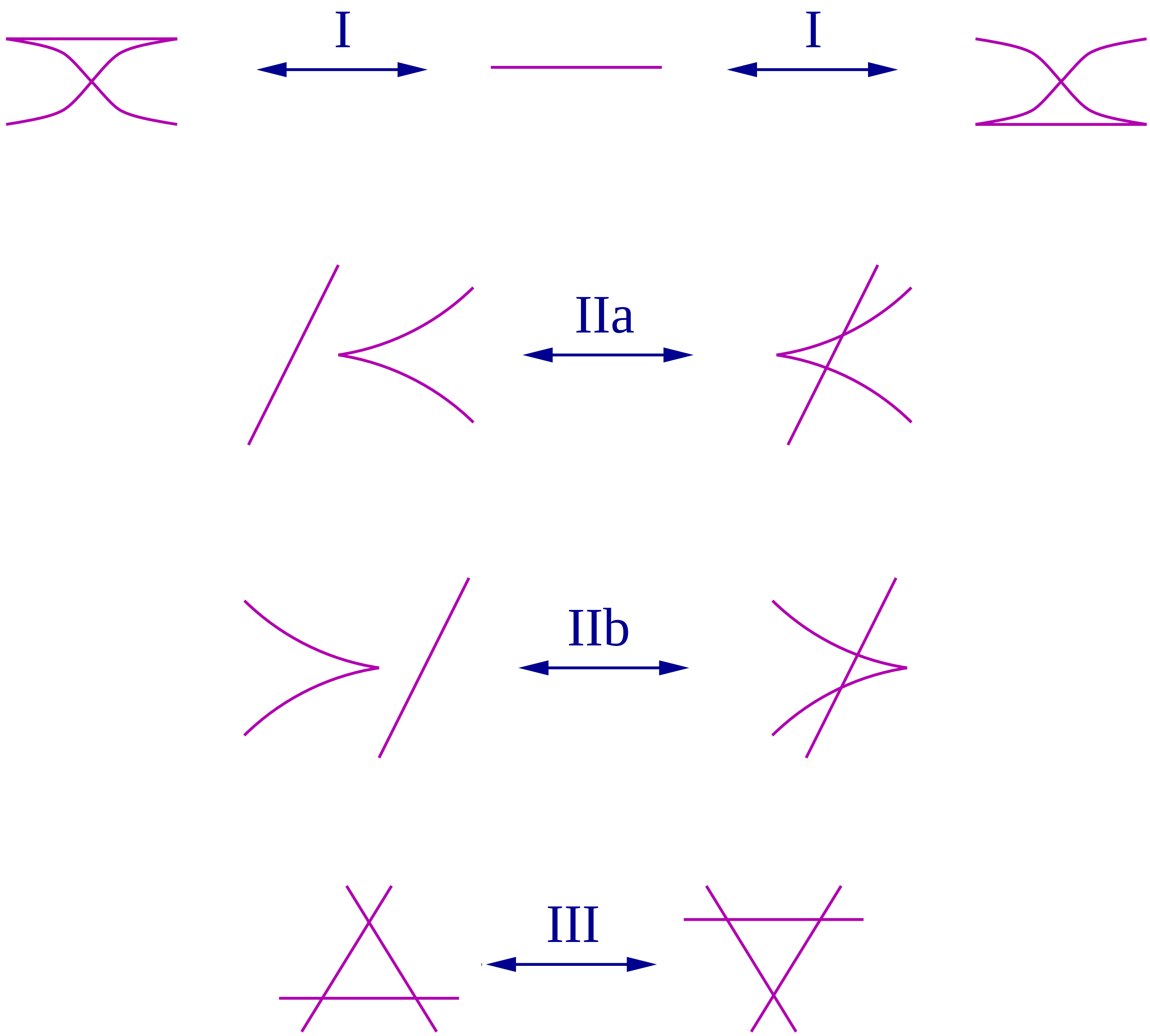


For front (xz) projection: since $y = dz/dx$, no vertical tangencies are allowed. Instead, use cusps to change direction in x.

No need to specify over/undercrossing information at crossings; the strand with the *lower* slope goes *over* the other strand.

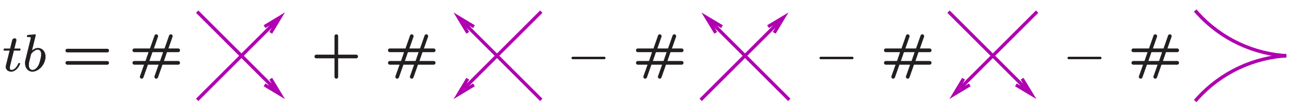


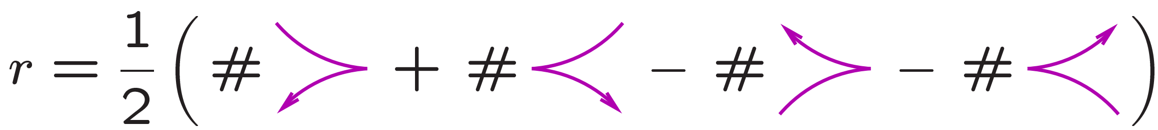
The "flying-saucer" unknot and the trefoil.

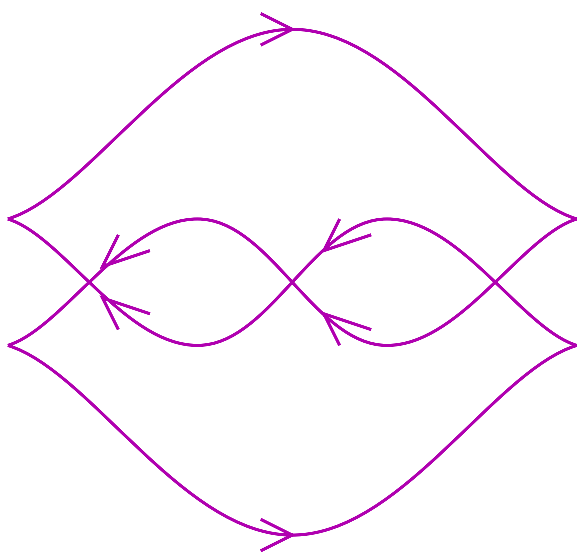


The Legendrian Reidemeister moves, relating the fronts of any Legendrian-isotopic knots.

Two “classical” invariants of Legendrian knots in standard contact $\mathbb{R}^3$: *Thurston-Bennequin number* and *rotation (Maslov) number*.







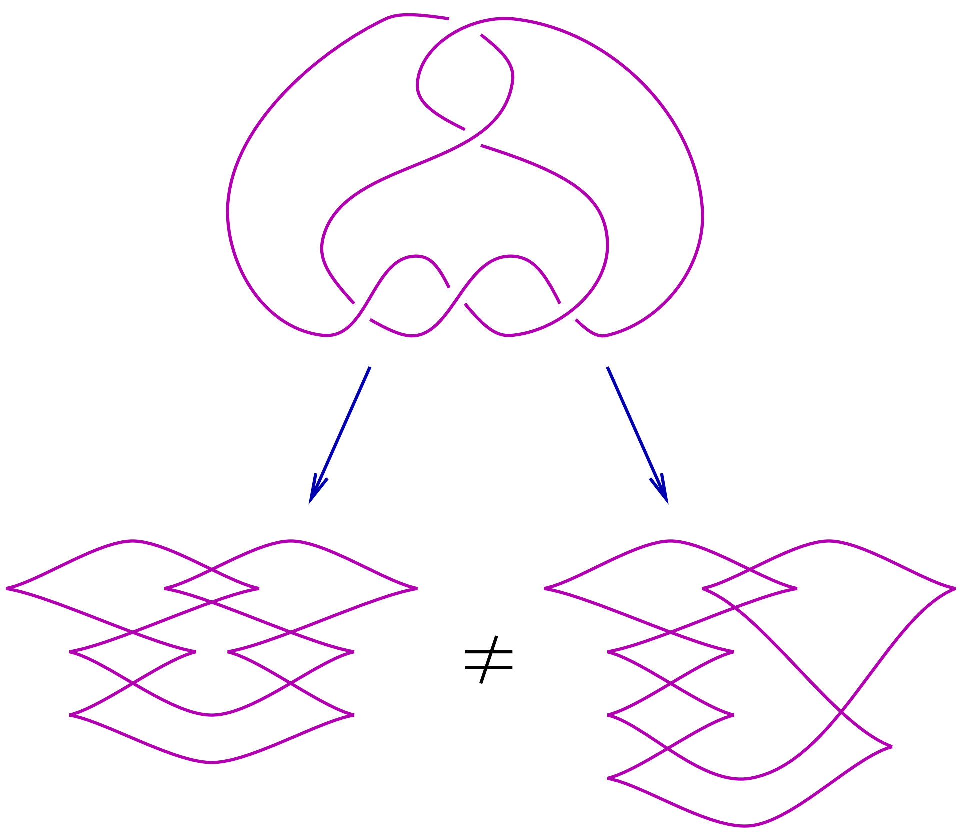
example: $tb = 1; r = 0$

tb, r form a *complete set of invariants* for some specific knots:

- unknot (Eliashberg-Fraser)
- torus knots, figure 8 knot (Etnyre-Honda)

Not true in general!

Chekanov, Eliashberg: 5_2 knot



New invariant uses *relative contact homology* (Eliashberg-Givental-Hofer).

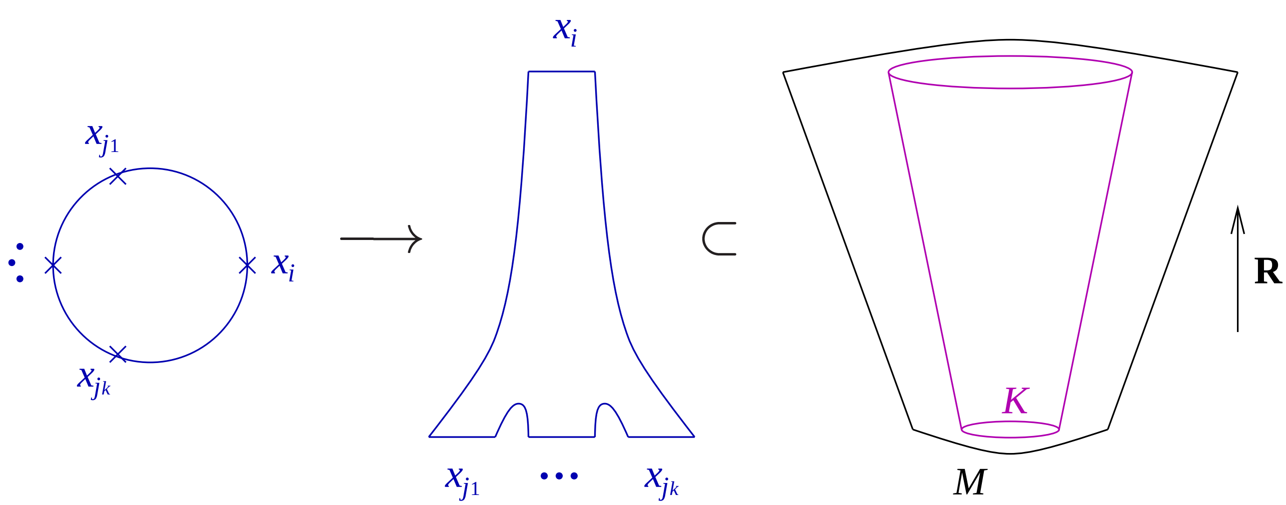
$M \times \mathbb{R} =$ symplectization of (M^3, α), with symplectic form $d(e^t\alpha)$, and a "nice" compatible almost complex structure.

$K \subset M$ Legendrian $\rightsquigarrow K \times \mathbb{R} \subset M \times \mathbb{R}$.

Label the Reeb chords for K by $x_1, \ldots, x_n$; for standard contact $\mathbb{R}^3$, the Reeb chords correspond to crossings in the Lagrangian projection of K.

Define A to be the free noncommutative algebra over $\mathbb{Z}/2$ generated by $x_1, \ldots, x_n$. The invariant is A, along with a grading and a differential ∂.

Define $\mathcal{M}^{x_i}_{x_{j_1}\cdots x_{j_k}}$ to be the moduli space of holomorphic maps from the punctured disk to $\mathbb{R}^3 \times \mathbb{R}$ of the form shown, where the punctures are mapped to Reeb chords $x_i, x_{j_1}, \ldots, x_{j_k}$, and the boundary of the disk is mapped to $K \times \mathbb{R}$.



Differential on A is given by Leibniz rule and

$$\partial x_i = \sum_{\dim \mathcal{M}=0+1} (\#) x_{j_1} \cdots x_{j_k}.$$

Contact homology still difficult to compute in general, but this *relative* case has a nice combinatorial interpretation, due to Chekanov.

Chekanov-Eliashberg differential graded algebra (DGA):
nonclassical invariant of Legendrian knots/links in standard contact $\mathbb{R}^3$

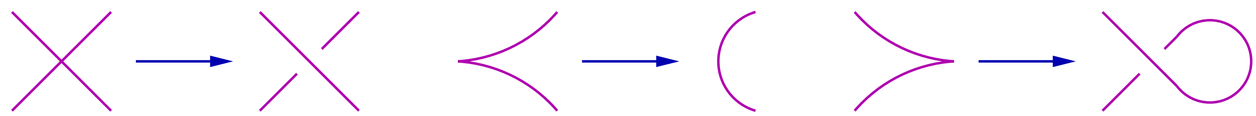
- Chekanov: original definition of DGA over $\mathbb{Z}/2$, graded over $\mathbb{Z}/(2r(K))$, in Lagrangian projection

- Etnyre, Sabloff, —: extension of DGA over $\mathbb{Z}[t, t^{-1}]$, graded over $\mathbb{Z}$

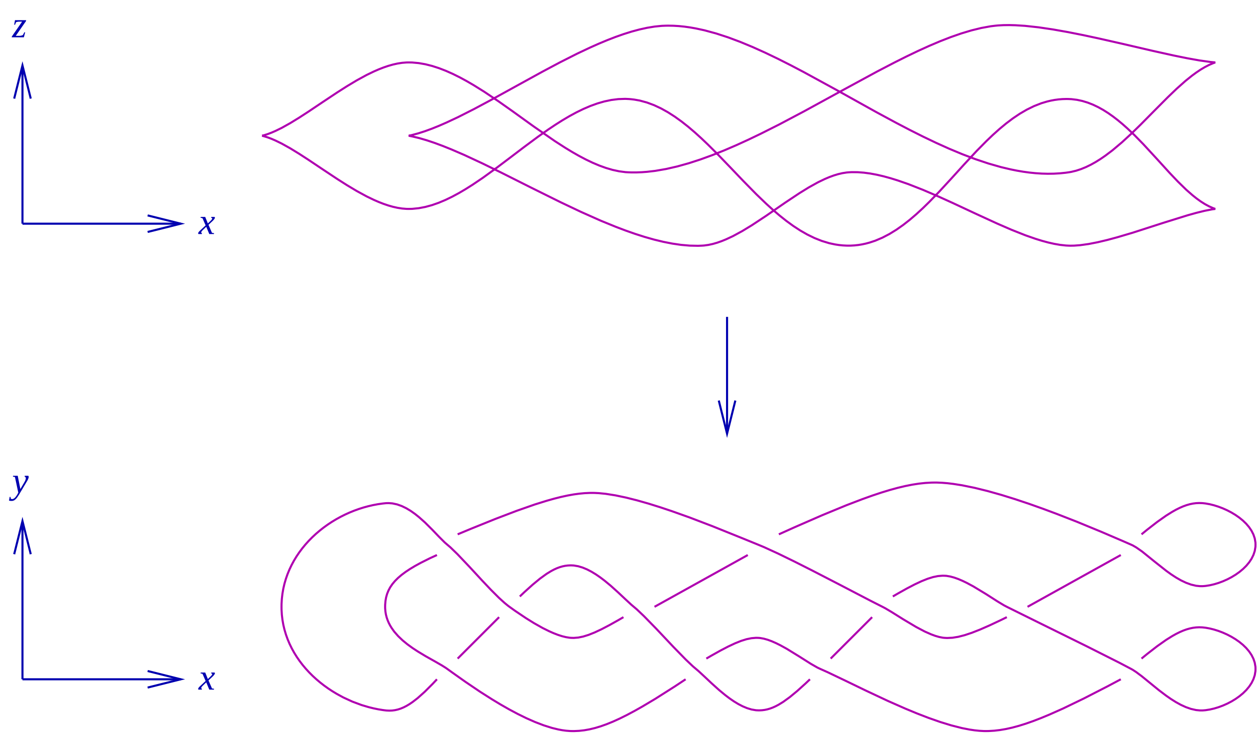
- —: reformulation of DGA in front projection

We define the DGA (A, ∂) for a "simple" front-projected Legendrian knot.

$A =$ free noncommutative algebra generated by *crossings* and *right cusps* of a front-projected Legendrian knot.

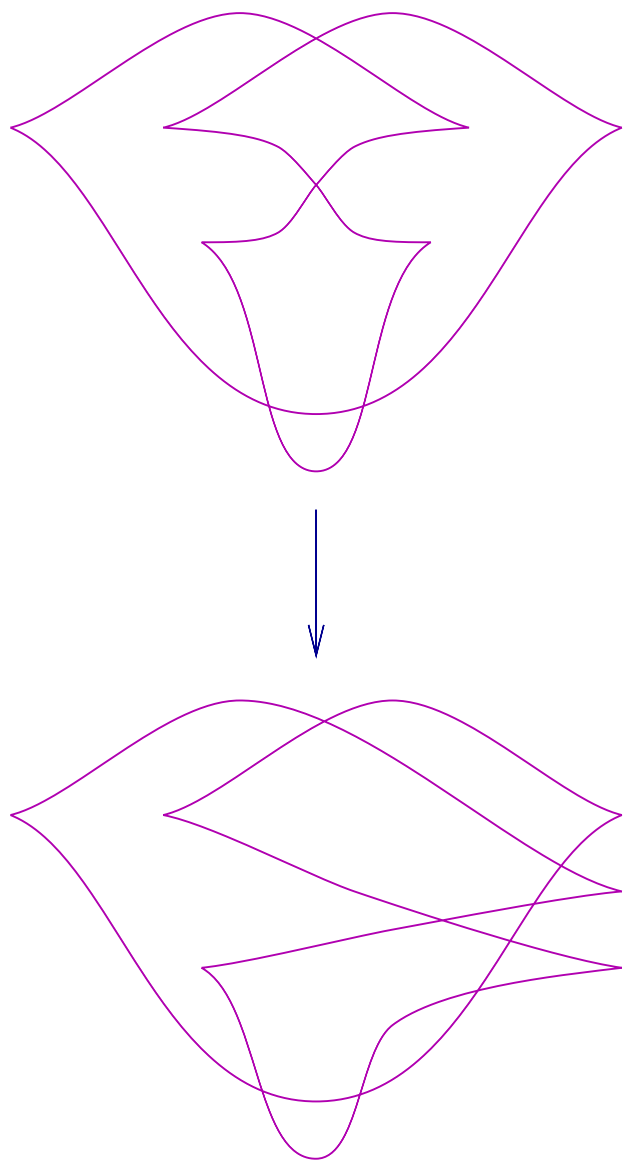


"Resolution" from front projection to
Lagrangian projection.



An example of resolution.

Simple front: a front, all of whose right cusps have the same x coordinate.

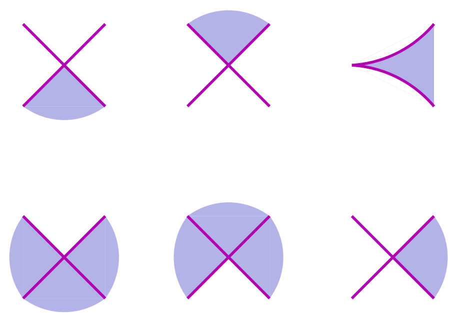


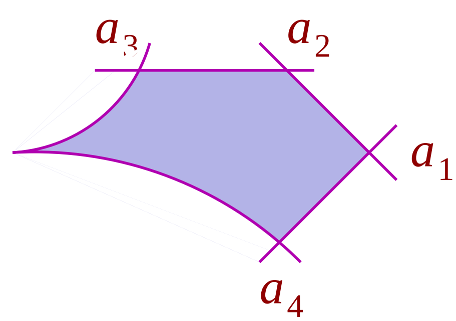
Turning a front into a simple front.

Differential ∂a_i counts embedded disks with boundary on the front and rightmost point at a_i.

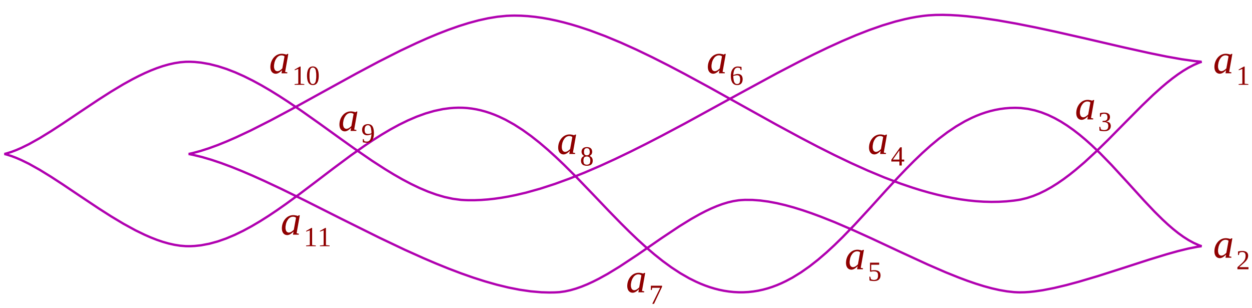
yes:

no:





$$\partial a_1 = \begin{cases} a_2 a_3 a_4 + \cdots & \text{if } a_i \text{ is a crossing} \\ 1 + a_2 a_3 a_4 + \cdots & \text{if } a_i \text{ is a right cusp} \end{cases}$$



Example:

$$\partial a_2 = 1 + a_3 + a_3 a_6 a_{10} + a_3 a_{11} a_7$$
$$\partial a_3 = 0$$
$$\partial a_6 = a_{11} a_8$$

Proposition $\partial^2 = 0$ and ∂ lowers degree by 1.

Two DGAs are *tamely isomorphic* if they are related by a series of "elementary automorphisms"

$$\begin{aligned} a_i &\mapsto a_i + (\text{not involving } a_i) \\ a_j &\mapsto a_j, \quad j \neq i. \end{aligned}$$

A *stabilization* of a DGA adds generators e_1, e_2 with $\partial e_1 = e_2$, $\partial e_2 = 0$. Two DGAs are *equivalent* if they are tamely isomorphic after some number of stabilizations to each.

Proposition Legendrian-isotopic knots have equivalent DGAs.

Problem: difficult to distinguish between DGAs in general.

Chekanov: Poincaré-type polynomials over $\mathbb{Z}/2$, using finite-dimensional "parts" of the homology of the DGA. But not effective in many cases.

Characteristic algebra: $A\,/\,\langle \partial a_1, \ldots, \partial a_n \rangle$. Incorporates Poincaré polynomial.

Proposition Legendrian-isotopic knots have isomorphic characteristic algebras, modulo adding generators with no relations.

Characteristic algebra can be used to answer *mirror question* of Fuchs and Tabachnikov: is there a Legendrian knot with $r = 0$ which is not Legendrian isotopic to its mirror (image under the contactomorphism $(x, y, z) \mapsto (x, -y, -z)$)? (Yes!)

DGA over $\mathbb{Z}/2$:

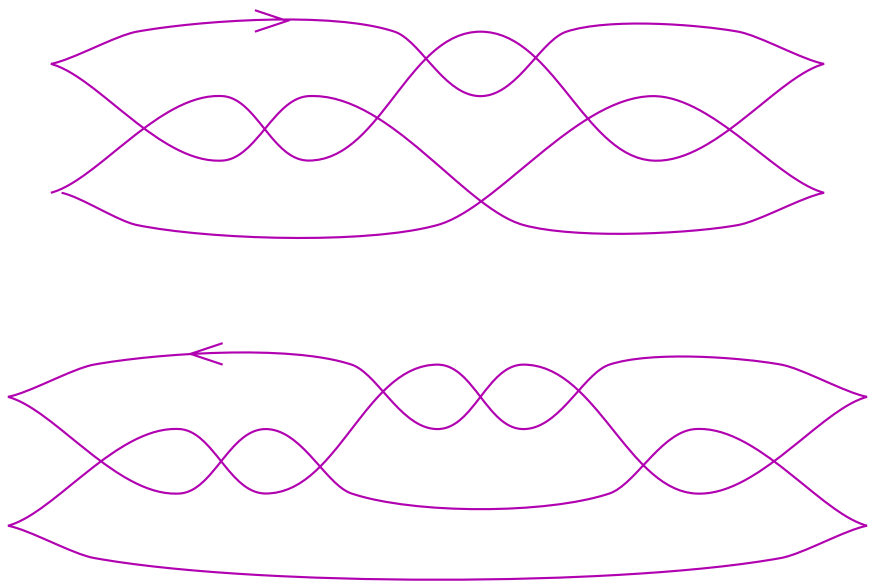
$$\begin{aligned}
\partial a_1 &= 1 + a_{10}a_5a_3 \\
\partial a_2 &= 1 + a_3 + a_3a_6a_{10} + a_3a_{11}a_7 \\
\partial a_4 &= a_{11} + a_5 + a_6a_{10}a_5 + a_{11}a_7a_5 \\
\partial a_6 &= a_{11}a_8 \\
\partial a_7 &= a_8a_{10} \\
\partial a_9 &= 1 + a_{10}a_{11}.
\end{aligned}$$

Characteristic algebra has relations

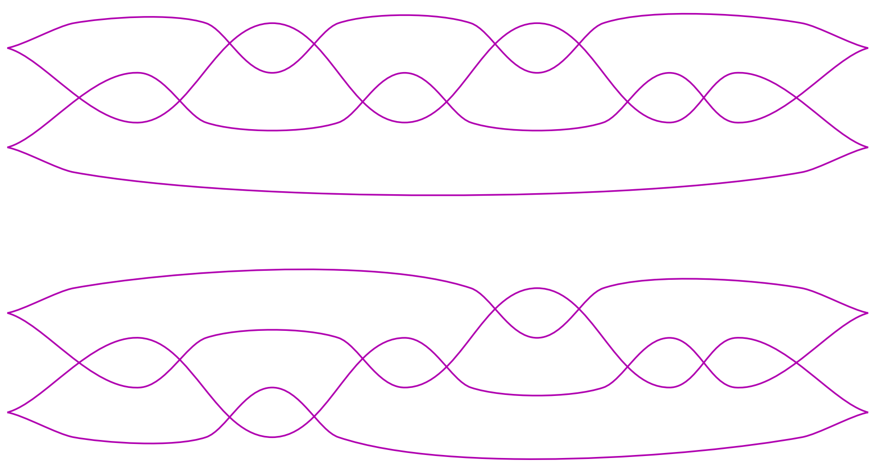
$$\begin{aligned}
1 &= a_{10}a_5a_3 \\
1 &= a_3a_{10}a_5 \\
1 &= a_{10}a_5 + a_6a_{10} + a_{10}a_5^2a_7 \\
1 &= a_{10}^2a_5^2 \\
(a_8 &= 0) \\
(a_{11} &= a_{10}a_5^2)
\end{aligned}$$

with grading: a_7, a_{10} degree 1; a_3 degree 0; a_5, a_6 degree -1.

Examples where the characteristic algebra is effective:

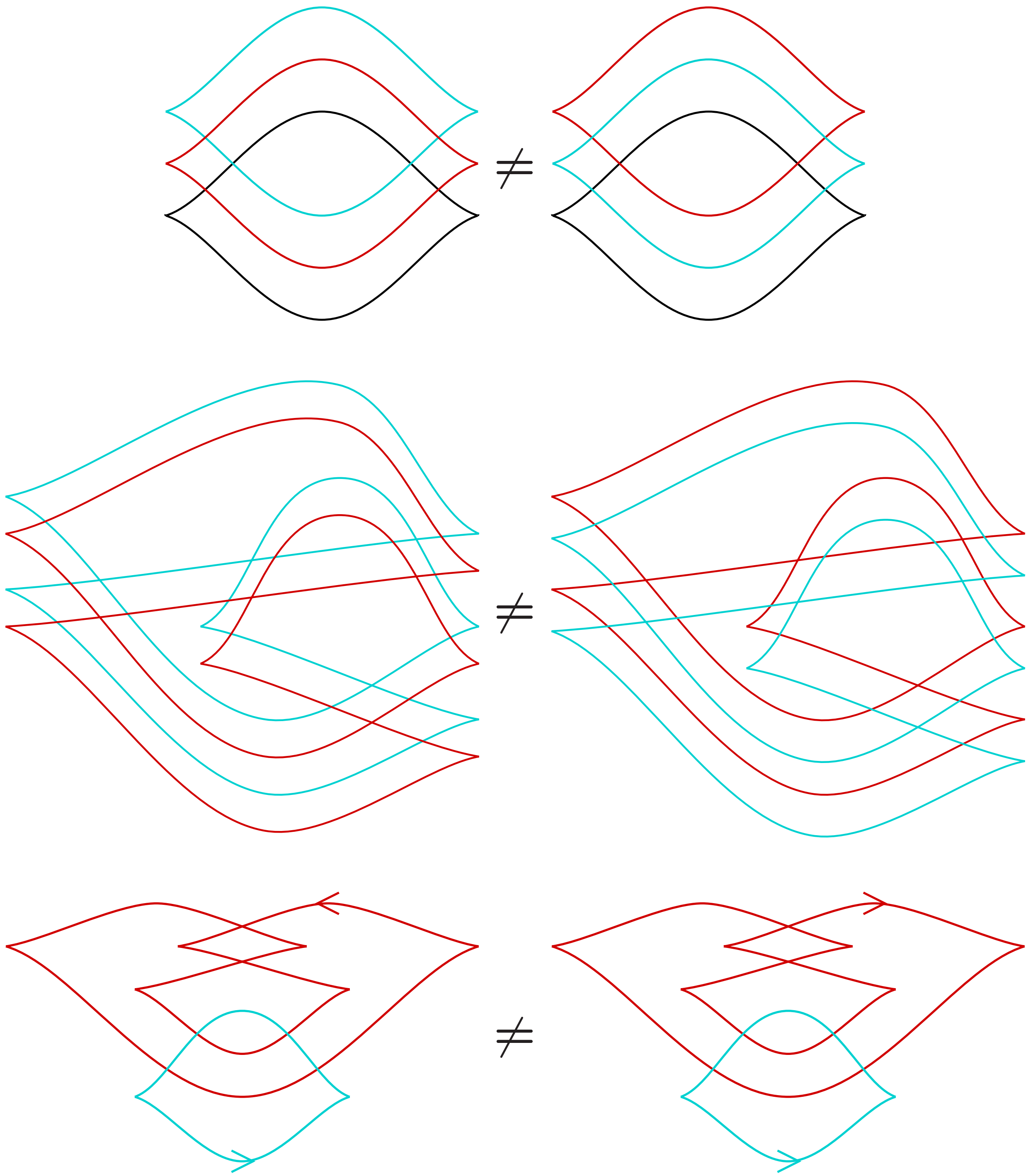


Two nonisotopic 6_3 knots: $r = 1, tb = -4$.



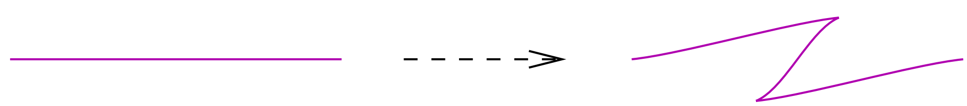
Two nonisotopic 7_2 knots: $r = 0, tb = 1$.

Michatchev: additional structure on DGA for *links*.



Questions

- Useful invariants for stabilized knots? DGA vanishes.



- Symplectic Field Theory?

- Nonclassical invariants of transversal knots?

- Extend to other contact manifolds? (Traynor, —: solid torus $S^1 \times \mathbb{R}^2$.)

- Computation of general contact homology?

- Relation of DGA to contact manifold obtained by Legendrian surgery?