QUANDLE HOMOLOGY & STATE SUM INVARIANTS

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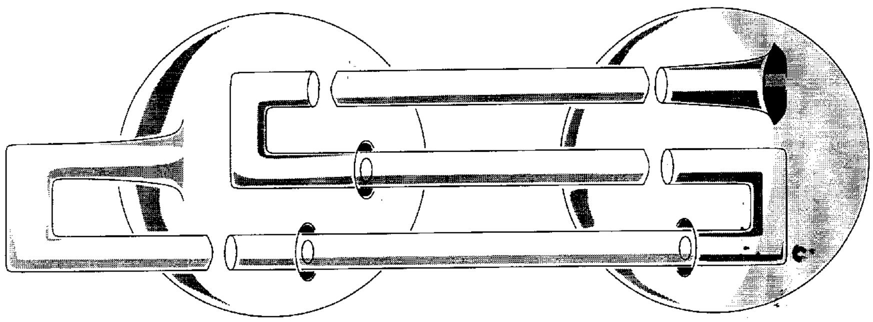
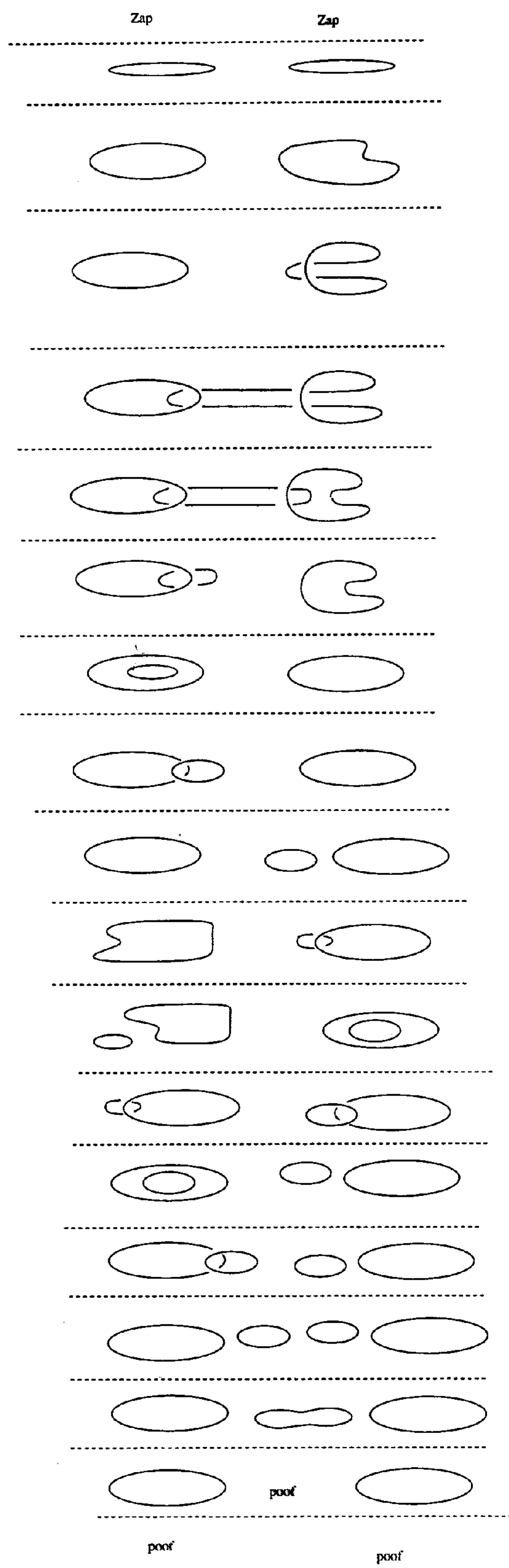
Example 10



$\Delta = 1-2T$


Zap
Zap
poof
poof
poof

Motivation

$$\underset{a\,b}{\cap} \rightarrow M_{ab} = \begin{array}{c} [\,0,\ iA,\ -iA^{-1},\ 0\,] \\ \scriptstyle 11 \quad 12 \quad 21 \quad 22 \quad ab \end{array}$$

$$\overset{a\,b}{\cup} \rightarrow M^{ab} = \begin{array}{cc} \begin{bmatrix} 0 \\ iA \\ -iA^{-1} \\ 0 \end{bmatrix} & \begin{array}{c} 11 \\ 12 \\ 21 \\ 22 \\ ab \end{array} \end{array}$$

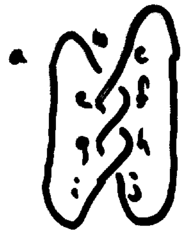
Little Calculation

$$a \bigcirc b \rightarrow \sum_{a,b} M_{ab} M^{ab} = [0,\ iA,\ -iA^{-1},\ 0] \begin{bmatrix} 0 \\ +iA \\ -iA^{-1} \\ 0 \end{bmatrix}$$

$$= -A^{2} - A^{-2}$$

 $\rightarrow R^{ab}_{cd} = A \left\langle \overset{a\,b}{\underset{c\,d}{|\ |}} \right\rangle + A^{-1} \left\langle \overset{a\,b}{\underset{c\,d}{\cup \atop \cap}} \right\rangle$

$$|^{a}_{b} \rightarrow \delta^{a}_{b} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{array}{cc} 11 & 12 \\ 21 & 22 \\ \multicolumn{2}{c}{ab} \end{array}$$

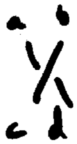


$$\sum M_{cd} M_{ab} R^{ab}_{ef} R^{ef}_{gh} R^{gh}_{ij} M_{a\ell} M_{aj}$$

$= \text{poly'n in } A, A^{-1}$

that specializes to Jones once write normalized.

$${}^{ab}\cup = \underset{ab}{\cap} = \underset{ab}{M} = M^{ab} = \begin{cases} 0 & a \neq b \\ 1 & a = b \end{cases}$$



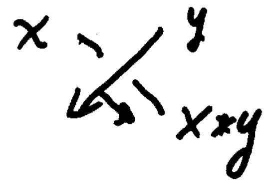
$$= R^{ab}_{cd} = \begin{cases} 1 & \text{if } a \neq b, d \quad b = c \\ & \quad\ \ d \neq c \\ 1 & \text{if } a = b = c = d \\ 0 & \text{otherwise} \end{cases}$$

eg  $= 1$,  $= 1$  $= 0$

 $= 1$  $= 1$

$\Sigma = 9$

A quandle, X, is a set with a binary operation $*$ such that



(I. idempotency)

$\forall a \in X, \quad a * a = a,$

(II. right-invertibility)

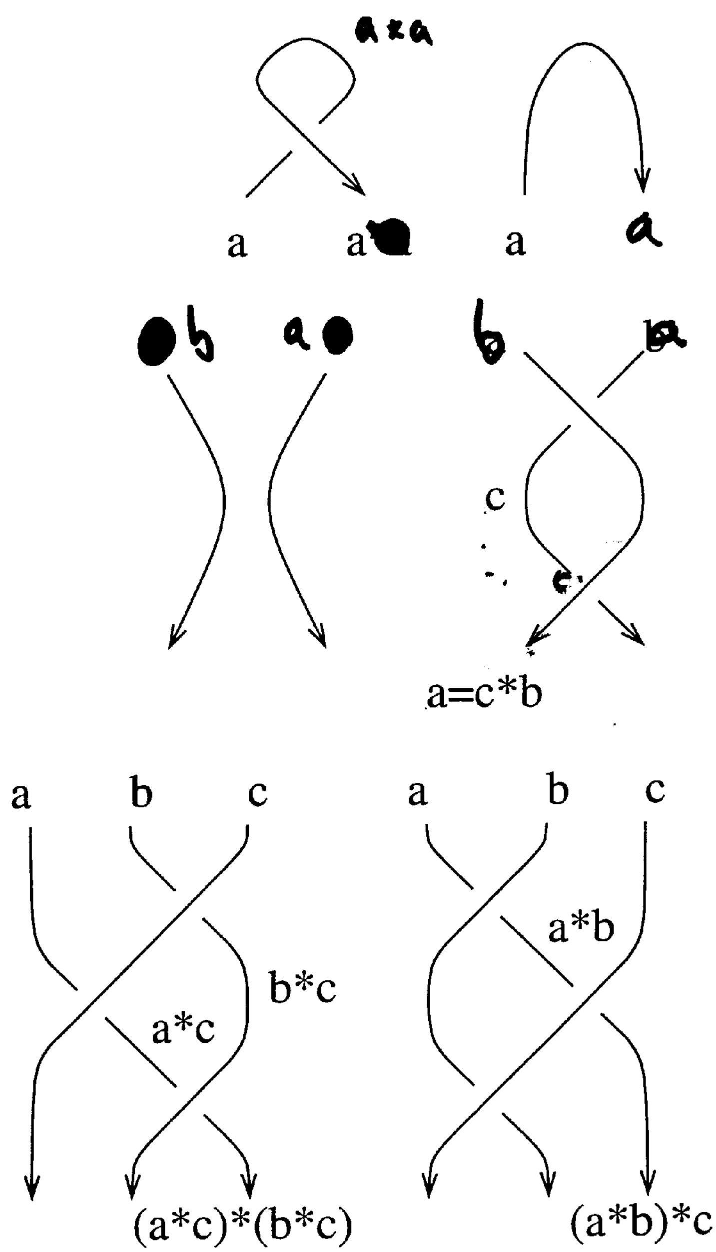
$\forall a, b \in X, \quad \exists! c \in X$

s.t. $a = c * b,$

(III. self-distributivity)

$\forall a, b, c \in X,$

$(a * b) * c = (a * c) * (b * c).$



a*a
a
a
a
a
b
a
b
a
c
a=c*b
a
b
c
b*c
a*c
(a*c)*(b*c)
a
b
c
a*b
(a*b)*c

1. Colors: Elements of a a finite quandle

   (a) A QUANDLE is a set, Q, with a binary operation $*$ defined such that

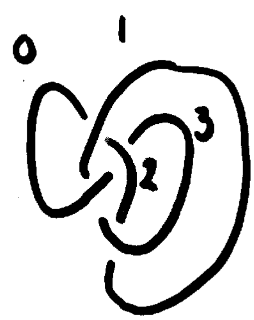
   - $a * a = a$
   - $\forall a, b \in Q \ \exists! c \in Q$ s.t. $a = c * b$
   - $(a * b) * c = (a * c) * (b * c)$

   (b) R_3 the 3-element dihedral quandle:

$$i * j = 2j - i \quad \mathrm{mod}(3)$$

   (c) S_4 a 4-element quandle:

$$\begin{array}{ccccccccc} 0 & = & 0*0 & = & 1*2 & = & 2*3 & = & 3*1 \\ 1 & = & 0*3 & = & 1*1 & = & 2*0 & = & 3*2 \\ 2 & = & 0*1 & = & 1*3 & = & 2*2 & = & 3*0 \\ 3 & = & 0*2 & = & 1*0 & = & 2*1 & = & 3*3. \end{array}$$



N.B. Quandle coloring is more gen than Fox coloring

Examples of quandles.

- *The trivial quandle* is any set X with $x * y = x$ for any $x, y \in X$.

- Any group G is a quandle $a * b = b^{-n} a b^{n}$ for $a, b \in G$.

- $QS(6) = \{(1234) = a, (1432) = A,$

$$(1243) = b, (1342) = B,$$

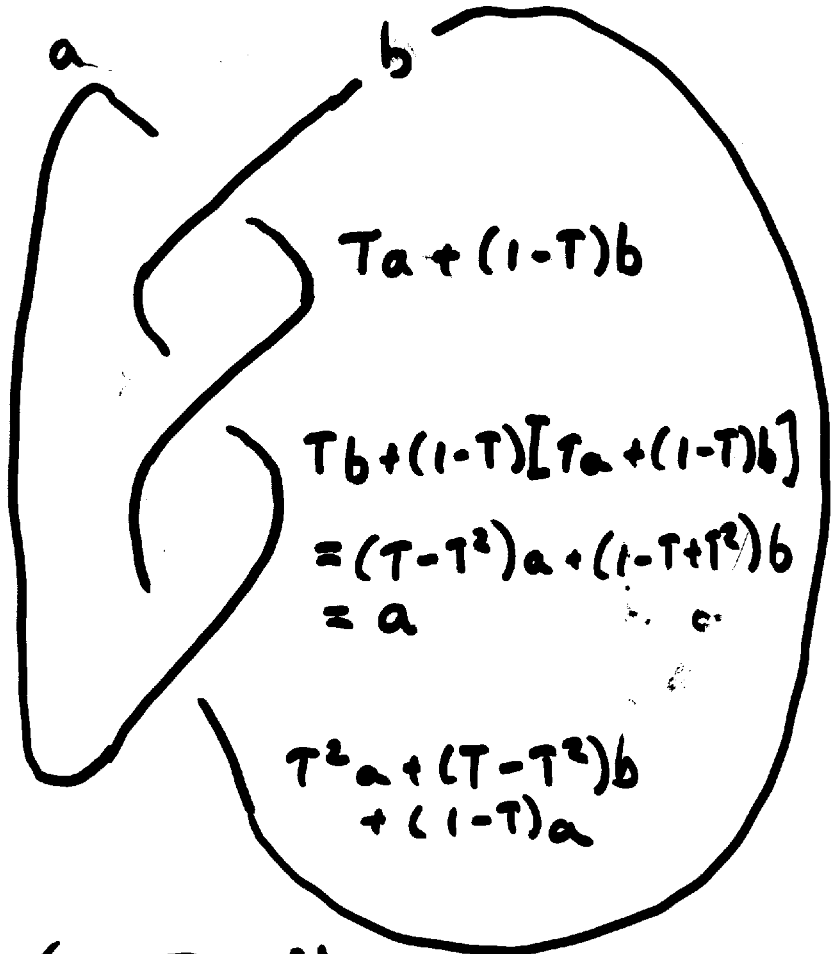
$$(1324) = c, (1423) = C\}$$

- *The dihedral quandle*, R_n: $i, j \in \{0, 1, \ldots, n-1\}$, $i * j = 2j - i$.

- *mod-n Alexander quandle*:

$$\mathbf{Z}_n[T, T^{-1}]/(h(T))$$

  with $h(T)$ a monic polynomomilop and $a * b = Ta + (1 - T)b$, $a, b \in M$.



$$(1 - T + T^2) = 0$$

Quandle Homology.

- $C_n^{\mathsf{R}}(X)$ is the f. a. g. gen. by $(x_1, \ldots, x_n)$ $x_j \in X$.

- $\partial_n : C_n^{\mathsf{R}}(X) \to C_{n-1}^{\mathsf{R}}(X)$:

$$\partial_n(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} (-1)^i$$

$$\Big[(x_1 * x_i, x_2 * x_i, \ldots, x_{i-1} * x_i, x_{i+1}, \ldots, x_n)$$

$$- (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)\Big]$$

- Let $C_n^{\mathsf{D}}(X)$ gen. by n-tuples $(x_1, \ldots, x_i, x_{i+1}, \ldots x_n)$

- If X is a quandle, then $\partial_n(C_n^{\mathsf{D}}(X)) \subset C_{n-1}^{\mathsf{D}}(X)$

- $C_n^{\mathsf{Q}}(X) = C_n^{\mathsf{R}}(X)/C_n^{\mathsf{D}}(X)$

We have a split seseq.

$$0 \to C_n^{\mathsf{D}}(X) \xrightarrow{i} C_n^{\mathsf{R}}(X) \xrightarrow{j} C_n^{\mathsf{Q}}(X) \to 0.$$

Thus there is a leseq:

$$\cdots \xrightarrow{\partial_*} H_n^{\mathsf{D}}(X;G) \xrightarrow{i_*} H_n^{\mathsf{R}}(X;G) \xrightarrow{j_*} H_n^{\mathsf{Q}}(X;G) \xrightarrow{\partial_*} \cdots$$

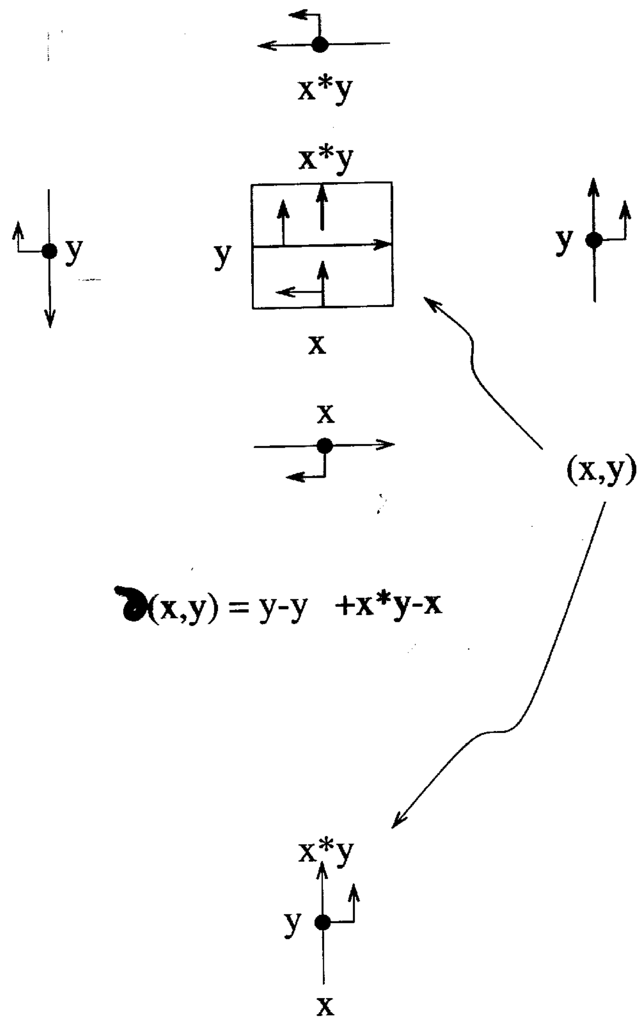
THM.

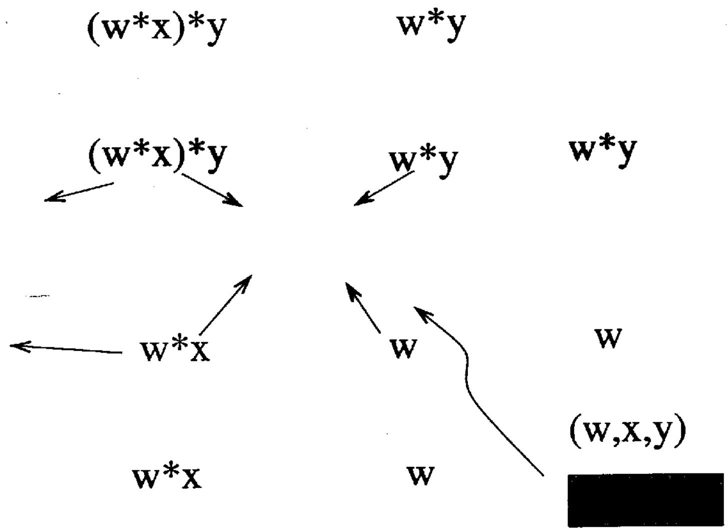
For $n = 3, 4$, the map

$$\partial_* : H_n^{\mathsf{Q}}(X) \to H_{n-1}^{\mathsf{D}}(X)$$

is trivial.

~~We'll give a geometric proof.~~


x*y
x*y
y
y
y
x
x
(x,y)
$\partial(x,y) = y-y \quad +x*y-x$
x*y
y
x



[redacted]

[redacted]

$$-\delta(w,x,y) = (w*x,y)-(w,y)-(w*y,x*y)+(w,x)$$

[redacted]

11 overlay

w*y

[redacted]

w

(**w*y**,x*y,z)

(w,y,z) [redacted]

w

w*z

[redacted]

(w,x,y)

[redacted] (w,x,y,z) [redacted]

-(w*z,x*z,y*z)

w

w*x

[redacted]

-(w,x,z)

[redacted]

-(w*x,y,z)

[redacted]

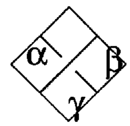
$$\partial(w,x,y,z)=(w*x,y,z)-(w,y,z)-(w*y,x*y,z)+(w,x,z)+(w*z,x*z,y*z)-(w,x,y)$$

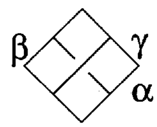
12 [illegible]

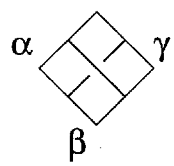
The cycle

$\alpha,\beta)+(\beta,\gamma)-(\beta,\alpha)$

$$\left[\begin{matrix} \alpha, \beta, \gamma \in R_3 \\ x * y = 2y - x \end{matrix}\right]$$







Remark.

Associated to the above diagram is a quandle.

It is given by generators and relators as

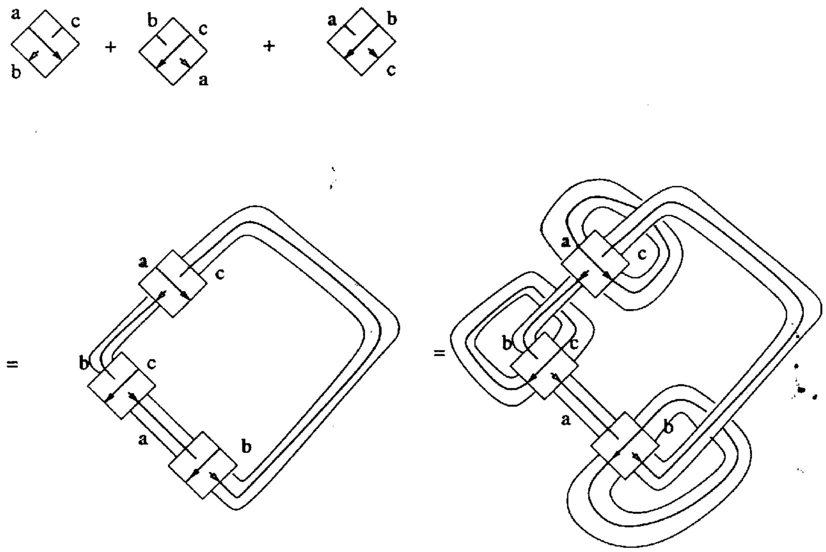
$$< x, y, z : x * y = z, y * z = x, y * x = z > .$$

It can be shown that this quandle is R_3.

$$H_2^Q(R_{2n+1}) = 0.$$

Thus the abstract knot is 2-homologically trivial.

We can show using shadow colors, that it is not 3-homologically trivial.


a
b
c
+
b
c
a
+
a
b
c
=
a
c
b
c
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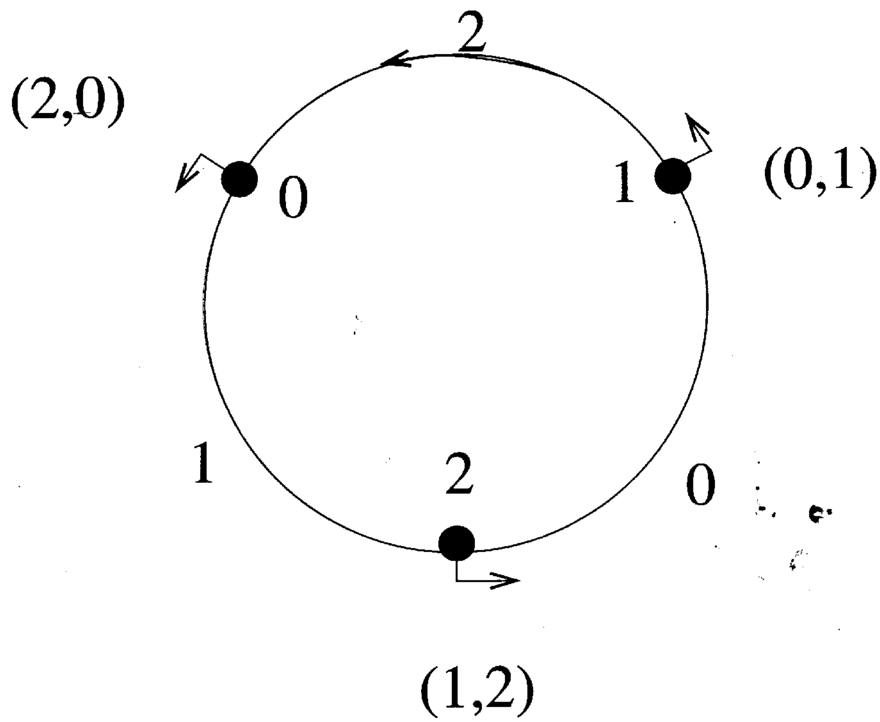
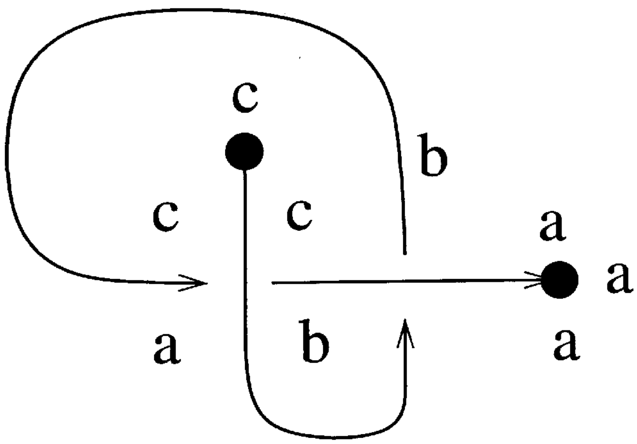
b
=
a
c
b
c
a
b



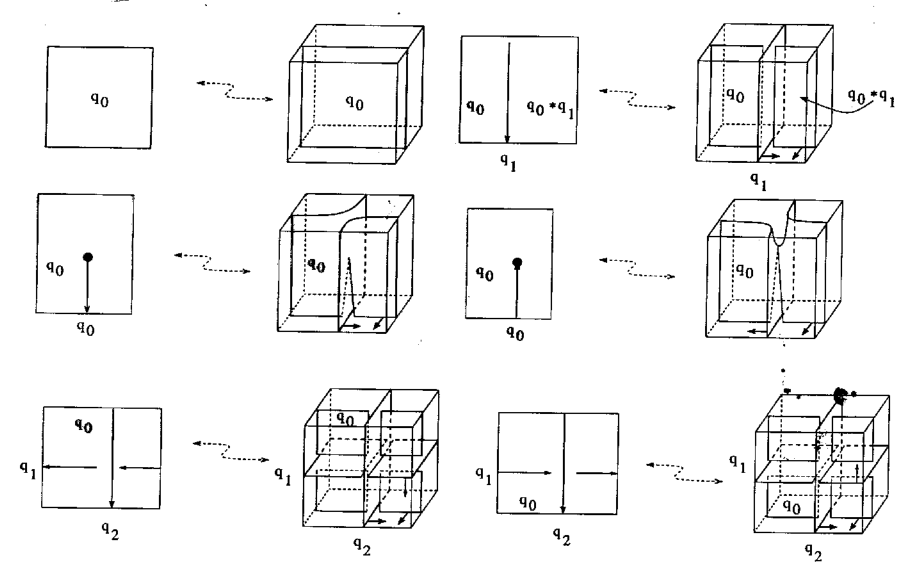
(0,1)+(2,0)+(1,2)

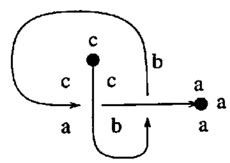
The example indicated here represent the 3-cycle $(a, b, c) + (a, c, a)$. One can show that this cycle generates

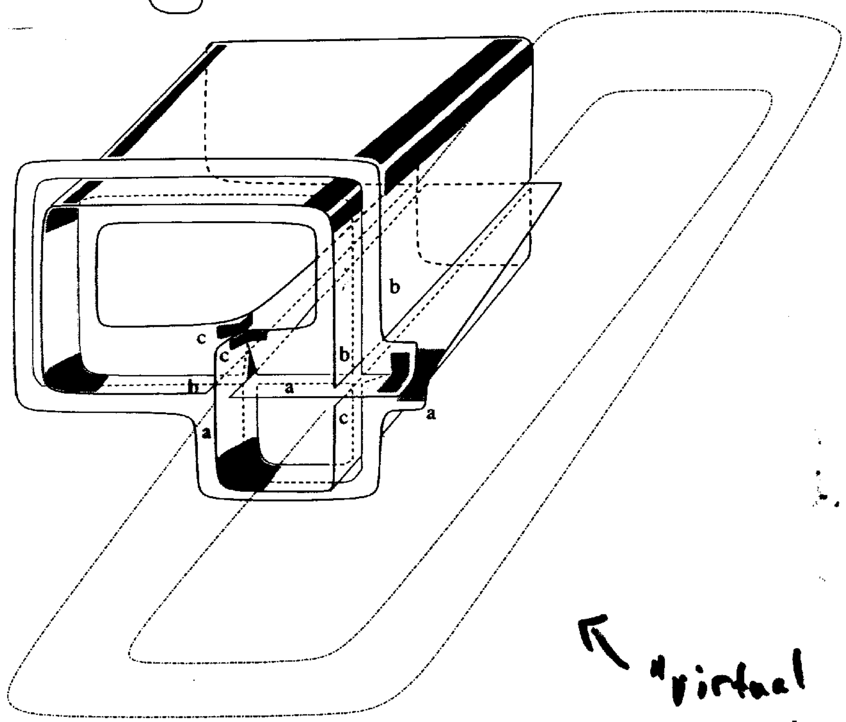
$$H_3^Q(R_3) = \mathbf{Z}/3$$



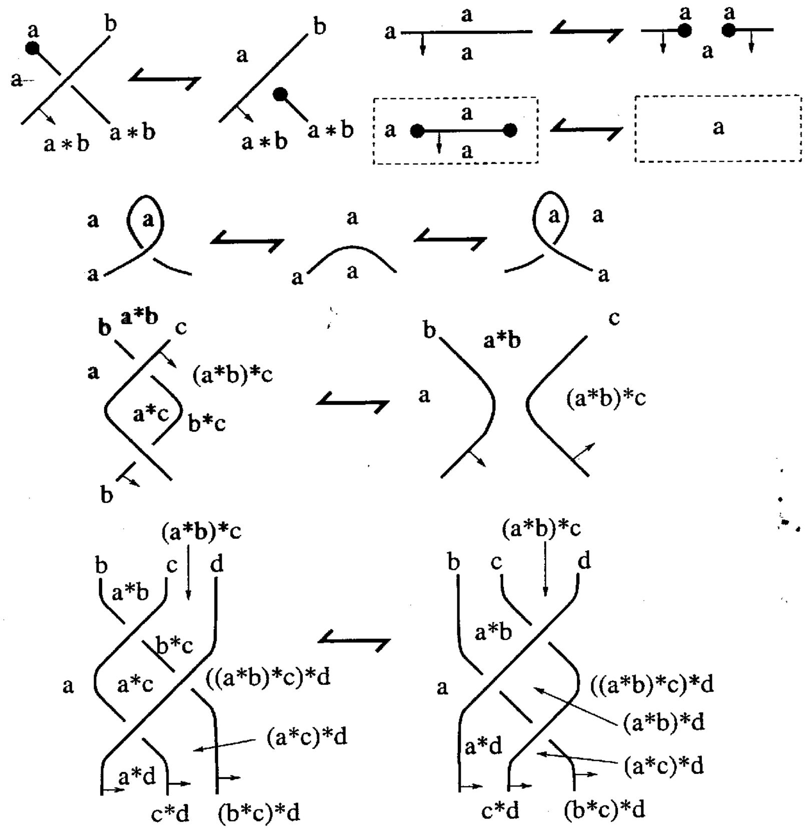
There is a method of going from a shadow colored classical diagram to a colored surface diagram.







"virtual
knotted
surface
diagram



a
b
a
a∗b
a∗b
a
b
a∗b
a∗b
a
a
a
a
a
a
a
a
a
a
a
a
a
a
a
a
a
b
a*b
c
a
(a*b)*c
a*c
b*c
b
b
a*b
c
a
(a*b)*c
(a*b)*c
b
c
d
a*b
b*c
a
a*c
((a*b)*c)*d
(a*c)*d
a*d
c*d
(b*c)*d
(a*b)*c
b
c
d
a*b
a
((a*b)*c)*d
(a*b)*d
a*d
(a*c)*d
c*d
(b*c)*d

Theorem.

$$\tau_2(3_1) \neq -\tau_2(3_1)$$

Notation:

- K is a classical knot
- $\tau_j(K)$ is the j twist spin of K
- $-\tau_2(K)$ is the same knot with the orientation reversed.

History:

- Fox (QT) Ex $10 = -$ Ex 11.

  LX $= (2-t)$ or $(1-2t)$.

- Ex. $12 = \tau_2(3_1)$. LX-module is non-prin. gen. by $= (2-t)$ & $(1-2t)$.

- Hillman, (1981) showed $\tau_2(3_1) \neq -\tau_2(3_1)$ using ~~L~~Levine pairing.
  Farber

- Ruberman (1983) used Casson-Gordan in-variants.

- $8_{17} \neq -8_{17}$ (Kawauchi, Hartley).

- FRS

Our Proof:

$$\exists \Phi(-) \in \mathbf{Z}[A].$$

Φ is an invariant of knotted surfaces. Here we use $A = \mathbf{Z}_3 = \langle t : t^3 = 1 \rangle$.

$$\Phi(\tau_2(3_1)) = 3 + 6t$$

&

$$\phi(-\tau_2(3_1)) = 3 + 6t^2$$

$\square$

Def of the knotted surface invariant Φ

Φ is a state-sum invariant:

So we define:

(1) a method of coloring knotted surface diagrams.

(2) a method of weighting the crossings (Triple points of the projection).

(3)

$$\Phi = \sum_{colors} \prod_{crossings} \text{Weight}$$

2. Weights: cocycles in the cohomology theory of quandles.

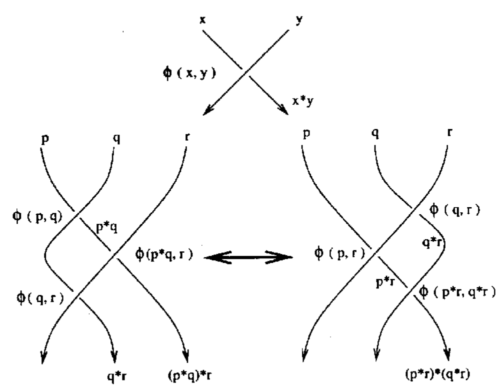
(a) n-cochains = h~~…~~) $\hom(C_n^Q(X), A)$

(b) $(\delta f)(x_0, \cdots, x_n) =$

$$\sum_{i=1}^{n} (-1)^{i-1} f(x_0, \cdots, \widehat{x}_i, \cdots, x_n)$$
$$+ \sum_{j=1}^{n} (-1)^{j} f(x_0 * x_j, \cdots, x_{j-1} * x_j, x_{j+1}, \cdots, x_n)$$

(c) 2-cocycles;

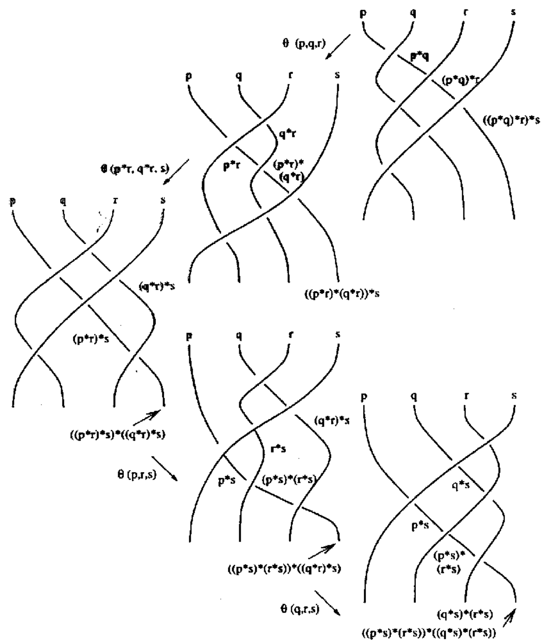
$$\phi(p, r) + \phi(p*r, q*r) = \phi(p, q) + \phi(p*q, r).$$

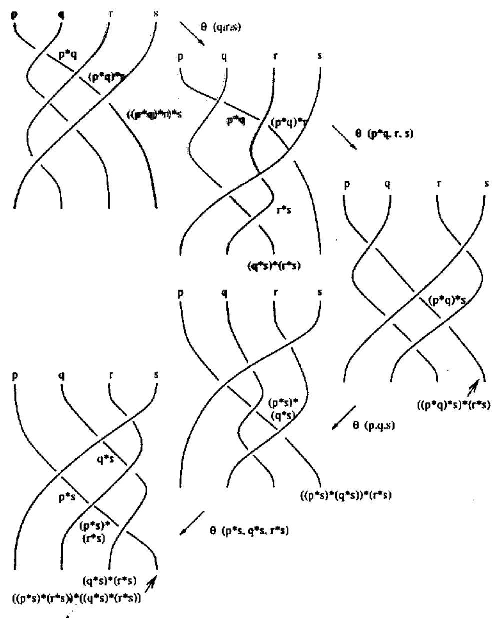


(d) 3-cocycles:

$$\theta(p,q,r) + \theta(p*r, q*r, s) + \theta(p,r,s)$$

$$= \theta(p*q, r, s) + \theta(p,q,s) + \theta(p*s, q*s, r*s).$$





(e) Assign 2-cocycles to crossings of classical knot diagrams. Assume $\phi(x,x) = 0$

(f) Assign 3-cocycles to triple points of knotted surface diagrams. Assume $\theta(x,x,y) = \theta(x,y,y) = 0$.

(g) If $f = \delta g$, then $\Phi_f = \#$ of colors.

$$\eta_1$$

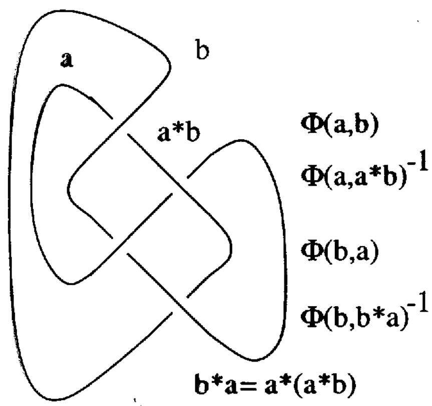
$$= -\chi_{(0,1,0)} + \chi_{(0,2,0)} - \chi_{(0,2,1)} + \chi_{(1,0,1)}$$

$$+\chi_{(1,0,2)} + \chi_{(2,0,2)} + \chi_{(2,1,2)}$$

$$[\eta_1] \neq 0 \in H^3(R_3; \mathbf{Z}_3)$$

$$[\eta_1] = [\chi_{(0,1)} + \chi_{(0,2)} + \chi_{(1,0)}$$

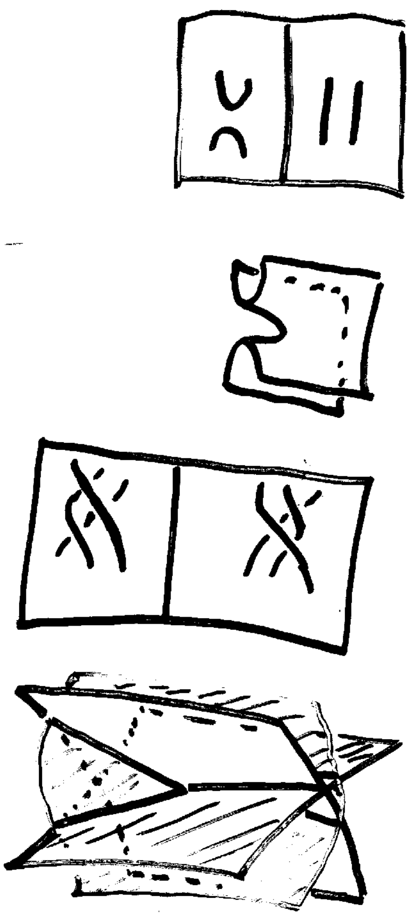
$$+\chi_{(1,2)} + \chi_{(2,0)} + \chi_{(2,1)}] \neq 0 \in H^2\cancel{(R_4; \mathbf{Z}_3)}\ (S_4; \mathbb{Z}/3\mathbb{Z})$$

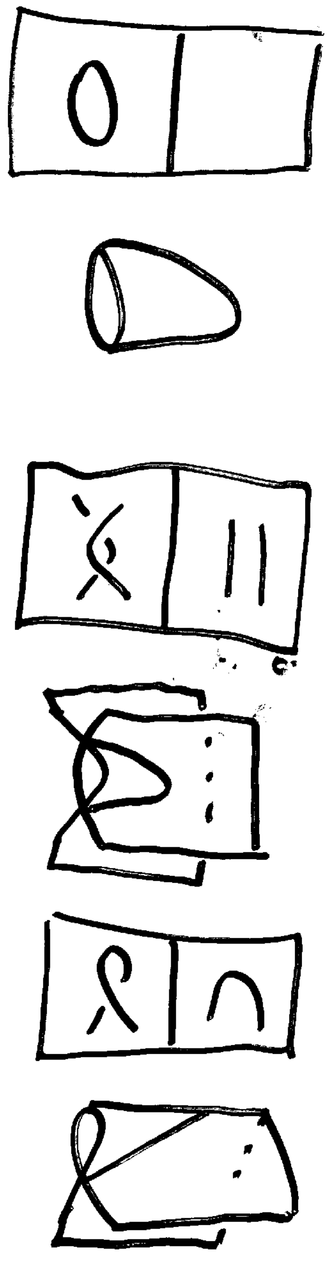


$$\sum_{a\,b} \Phi(a,b)\ \Phi(a,a*b)^{-1}\ \Phi(b,a)\ \Phi(b,b*a)^{-1}$$

$\phi = \eta_2$

$4 + 12t$





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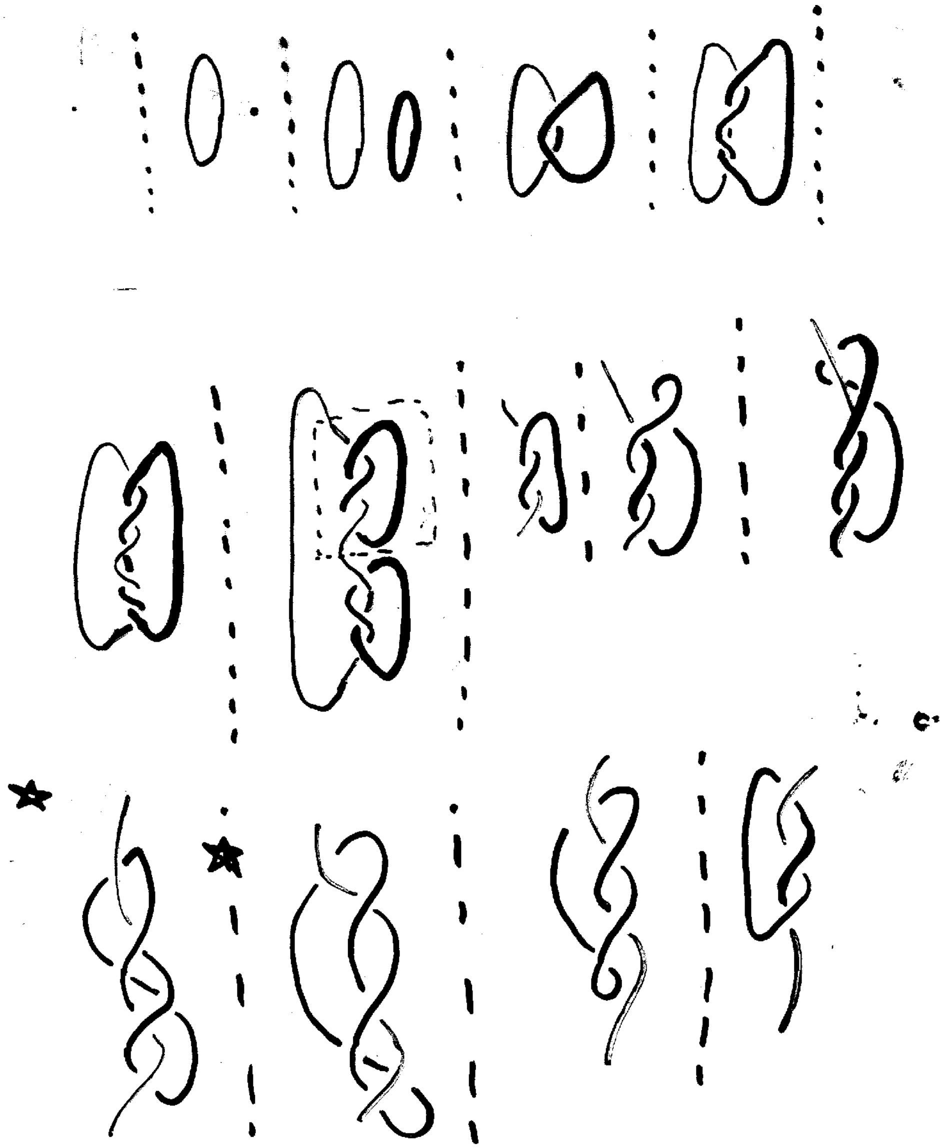


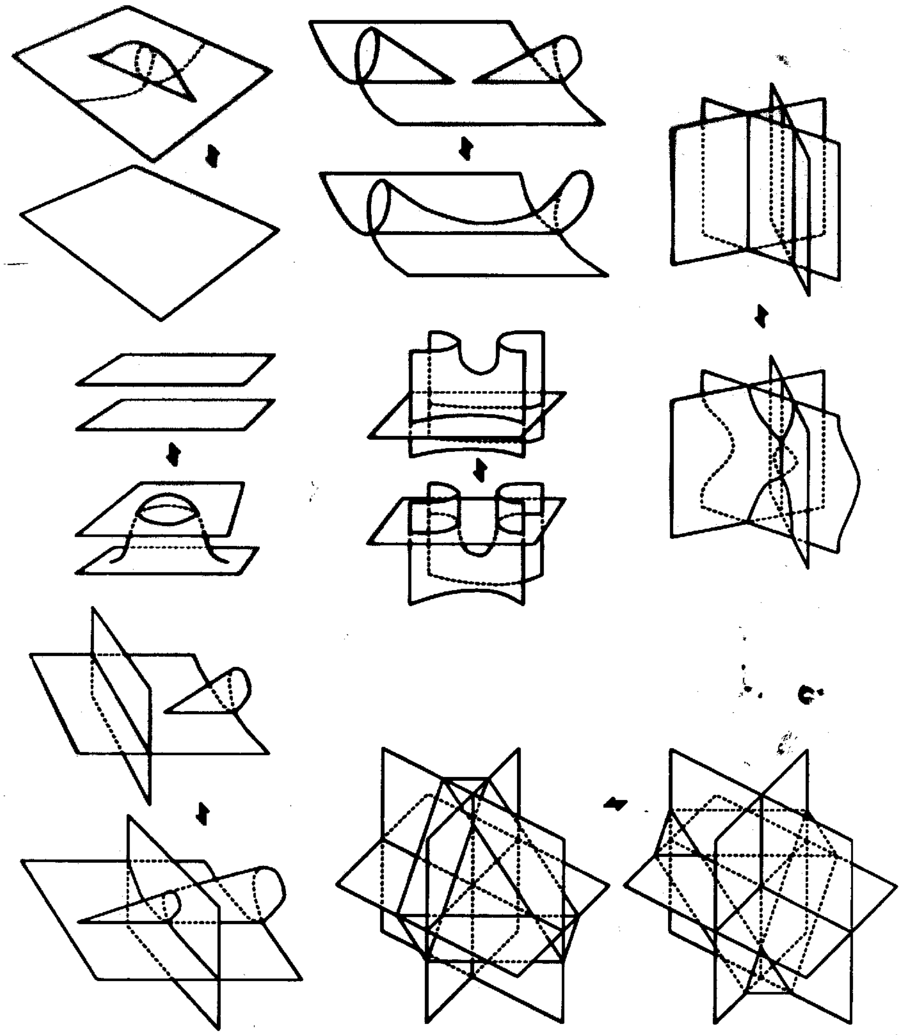


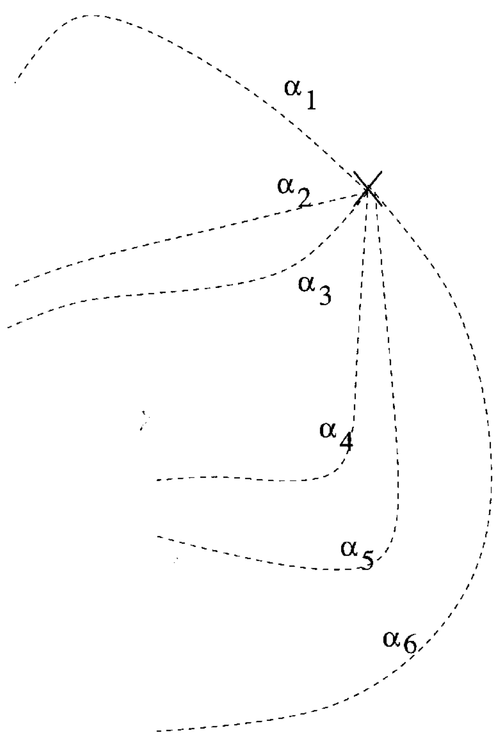
Figure 7: Roseman Moves.

Quandle of $\tau_2(3_1)$

$$= \langle x_1, x_2 | \quad x_2 = x_1 * (x_2 x_1), \quad x_2 = x_2 * (x_1^2) \rangle$$

The quandle $-\tau_2(3_1)$ is

$$= \langle x_1, x_2 | \quad x_2 = x_1 * (x_2 x_1), \quad x_2 = x_2 * (x_1^2) \rangle.$$



Recent Stuff:

1. 2-cocycles as obstructions to extensions

2. Ditto 3-cocycles

3. Constructions of cocycles in case of Alexander quandles

4. Structure of Dihedral quandles R_n, n not prime.

5. Twisted cohomology
more cocycles still!

Questions

1. $H^Q_*(\Sigma_n)$, Σ_n symmetric group.

2. $K \rightsquigarrow$ Quandle(K)
$[K] \neq 0 \in H_2(\text{Quandle}(K))$
?/