Lagrangian fibrations with Lefschetz* and circle fibers

or

Completely integrable systems with focus-focus* and elliptic singularities

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* a.k.a. <u>nodal</u>, fishtail

Lagrangian fibrations

Def'n: (M^{2n}, ω) symplectic

$\downarrow \pi$ locally trivial fibration

B^n

$\omega|_{\pi^{-1}(b)} = 0$

for all $b \in B$

Ex: $(T^n_x \times T^n_y, dx \wedge dy)$

$\pi \downarrow$

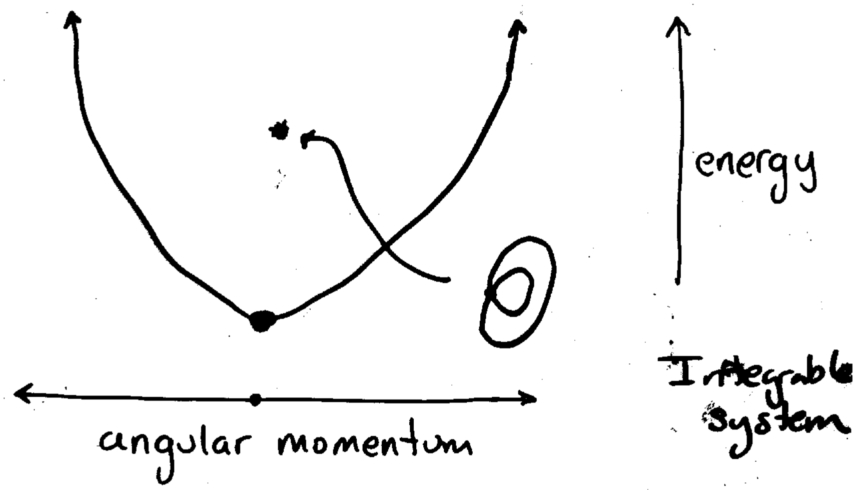
T^n $\quad \pi(x,y) = x$

Fact: $\pi^{-1}(b) = T^n$ if compact, no bdy

(Arnold-Liouville)

Example from mechanics:

Spherical pendulum



Duistermaat: no global action-angle coordinates; monodromy around singular fiber * is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Example from algebraic geom.:

K3 surface

= degree 4 hypersurface in $\mathbb{C}P^3$

$$\begin{array}{c} K3 \\ \pi \downarrow \\ \mathbb{C}P^1 = S^2 \end{array}$$

(Lefschetz) holomorphic elliptic fibration

$\pi^{-1}(p)$ = elliptic curve (T^2)
holomorphic

for <u>generic</u> p.

24 singular fibers 

monodromy $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

How is this Lagrangian?

K3 is hyper-Kähler

$(K3, J_i, \omega_i) \quad i = 1, 2, 3$

$$\left.\begin{array}{r} a J_1 + b J_2 + c J_3 \\ \text{with} \quad a^2 + b^2 + c^2 = 1 \end{array}\right\} \begin{array}{l} \text{an } S^2 \\ \text{of} \\ \text{complex} \\ \text{structures} \end{array}$$

Take holomorphic fibration with respect to J_1, ω_1

$\rightsquigarrow$ then Lagrangian with respect to ω_2.

Examples from toric geometry:

$(M^4, \omega) \circlearrowleft T^2$ Hamiltonian torus action

closed mfld

(effective)

moment map

$\mu : M \to \mathbb{R}^2 =$ dual of Lie Alg. of T^2

Atiyah - Guillemin - Sternberg:

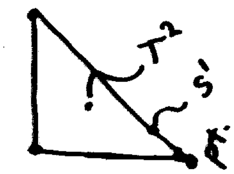
$\mu(M)$ is a convex polygon,

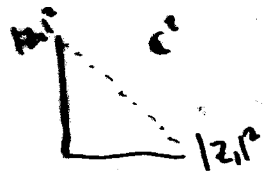
convex hull of images of fixed pts.

Delzant: $\mu(M)$ classifies (M, ω)

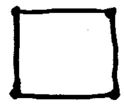
up to equivariant symplectomorphism

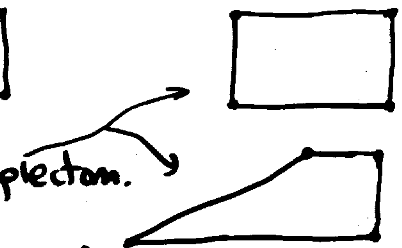
$\mathbb{C}P^2$:





$S^2 \times S^2$:





symplecto.
(different torus actions)

"The right pictures"

Why? (M^{2n}, ω) singular Lagrangian fibration

$\downarrow \pi$

$B^n \supset B_0$ = image of regular fibers

$\pi^{-1}(b)$ compact, no bdy.

$\boxed{B_0 \text{ has an integral affine structure}}$

= action coordinates, locally.

Philosophy: Look downstairs to understand the topology upstairs.

- Can build B_0 by gluing coordinate patches with maps from

$$SL(n,\mathbb{Z}) \ltimes \mathbb{R}^n$$

Restrict to: dim. 4 $(n=2)$

and allow only nodal, circle and point singular fibers

Zung: Base B, with the affine structure on B_0, determines (M,ω) whenever $B - B_0 \neq \emptyset$.

Draw pictures in dim 2 ...
then what?

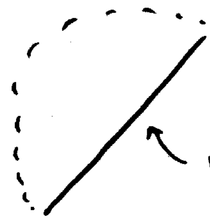
Realization: Say what smooth 4-manifolds admit such a Lag. fibration.

Rational blowdowns: Fiber the rational balls used in generalized rational blowdowns ⇒ a smooth surgery can be done symplectically

Space of such fibrations on a 4-manifold: Connect the different toric fibrations with a path of fibrations.

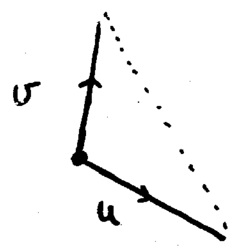
Neighborhoods of...

circle fiber



rational slope

point fiber

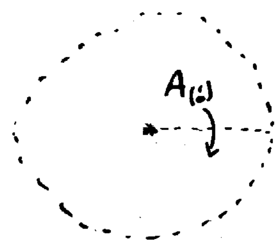


"symplectically equivalent if primitive integral vectors u, v span $\mathbb{Z}^2$

(i.e. $u \times v = 1$)

Otherwise an **orbifold**!

nodal fiber



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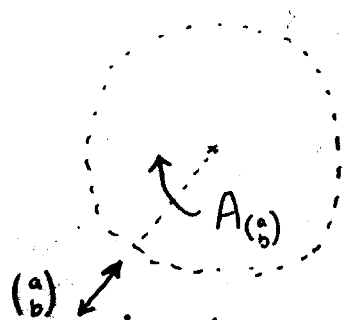


eigenline

vanishing cycle of the nodal fiber is $\binom{-b}{a}$

(eigenvector of $A_{\binom{a}{b}}^{-T}$)

cut open, reglue using

$$A_{\binom{1}{0}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

or

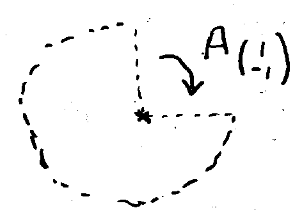
cut open along $\binom{a}{b}$

reglue using

$$A_{\binom{a}{b}} = \begin{pmatrix} 1-ab & a^2 \\ -b^2 & 1+ab \end{pmatrix}$$

$(a,b)=1$

or even



Lemma:  symplectomorphic

Gromov: A symplectic 4-manifold bounded by a standard 3-sphere is a standard ball (or its blow-up).

no exceptional curves here...

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move nodal fiber images in B:

along eigenlines → symplectom.

off of eigenlines → deformation of ω

(change $[\omega]$)
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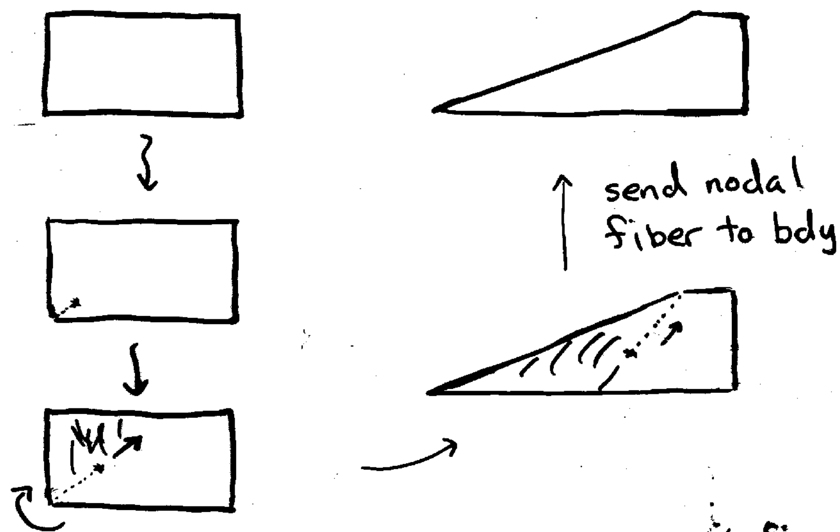
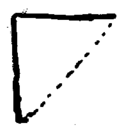
A family of fibrations:



$A_{(1,1)}$

Apply $A_{(1,1)}^{-1}$ to the triangle;

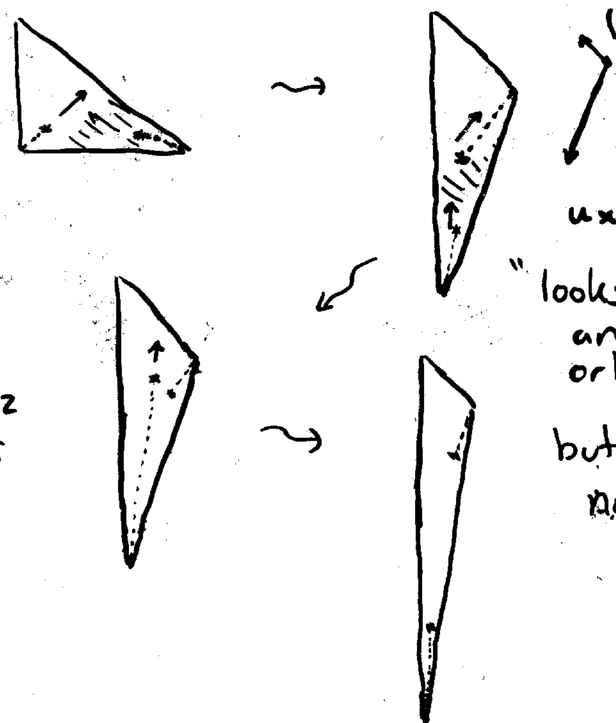
$$A_{(1,1)}^{-1} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$$



switch
direction
of branch
cut.

One topology - many diagrams

Other images of $\mathbb{C}P^2$...



$\begin{pmatrix}-1\\1\end{pmatrix} = u$

$\begin{pmatrix}-1\\-3\end{pmatrix} = v$

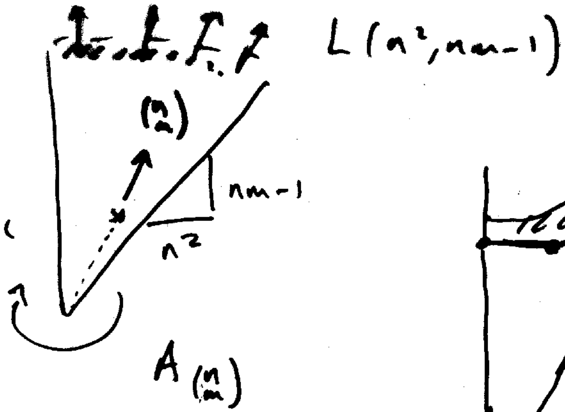
$u \times v = 4$

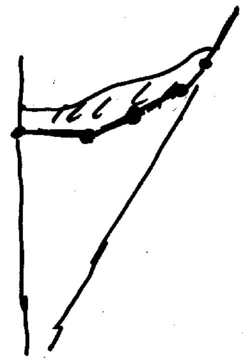
"looks like an orbifold"

but its not!

Hurewicz moves

"braiding of nodal images"





boundary is ω-convex.

vector field X

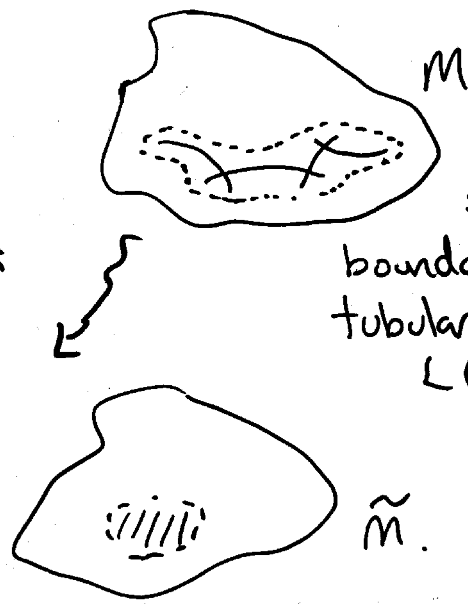
$\pitchfork$ to bdy and such

$\mathcal{L}_X \omega = \omega$.

<u>Thm</u> The **generalized rational blowdown** surgery, applied to a symplectic 4-manifold (along symplectic spheres), yields a symplectic manifold.

<u>Def</u>

remove nbhd of spheres, glue in $B_{n,m}$



union of spheres, boundary of tubular nbhd is $L(n^2, nm-1)$.

Fintushel + Stern $m=1$

Park $m>1$

Thm (–) m=1 case.

Consequences:

– Symplectic structures on interesting smooth 4-manifolds:

- Infinite family not homotopic to a complex surface (Fintushel + Stern)
- Exotic K3's (homeomorphic to K3, not diffeomorphic) (Gompf + Mrowka)

m>1

- Other interesting 4-mflds? (Park)
- Unexpected symplectic spheres in $\mathbb{CP}^2 \# n\overline{\mathbb{CP}}^2$??

Realization

" <u>Thm</u> A singular Lagrangian fibration* with nodal and circle fibers exists on

- K3
- $S^2 \times S^2$, $\mathbb{C}P^2 \# n\overline{\mathbb{C}P}^2$ — toric examples
- $T^2 \times S^2 \# n\overline{\mathbb{C}P}^2$
- T^2 bundles over T^2 with monodromy $\left(\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}, I\right)$."

Geiges

* Assume B orientable for convenience

Zung: Base B, when orientable, must be S^2, D^2, cylinder or T^2.

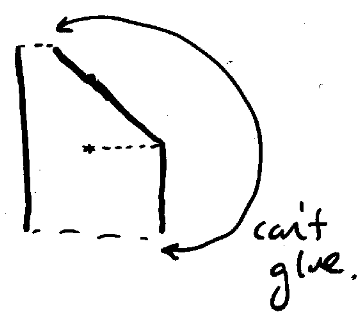
<u>Why?</u>

Nodal fibers introduce curvature.

B has no bdy: each nodal fiber contributes $1/12$ to $\chi(B)$

$$\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\right)^6 = I$$

B has bdy: nodal fibers make lines converge:



So...

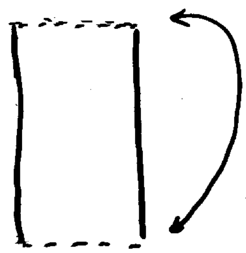
$B = T^2 \rightsquigarrow$ no singular fibers

T^2 bundles over T^2

$B = S^2 \rightsquigarrow$ 24 nodal fibers

$\Rightarrow K3$

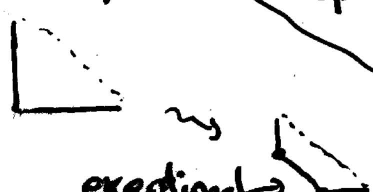
$B =$ cylinder



$T^2 \times S^2$

$= S^1 \times (S^1 \times S^2)$

usual blow-up:



exeptional sphere S, $S \cdot S = -1$.

blow up $\rightsquigarrow$

$T^2 \times S^2 \# \overline{\mathbb{C}P}^2$

$B = D^2$ Idea: push all nodal fibers to the boundary (of B).

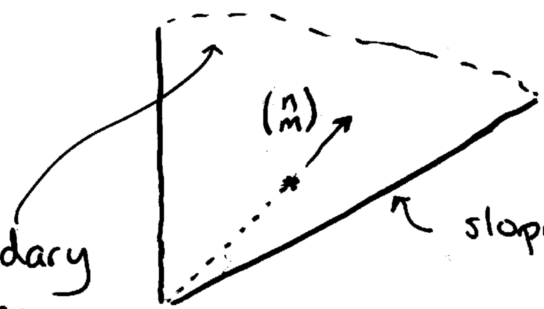
"See a toric manifold."

Uh oh... might look like an orbifold.

Need to bring to a standard position... "unbraid" along eigenlines.

Example from smooth topology:

rational balls $B_{n,m}$



boundary is lens space $L(n^2, nm-1)$

$n, m \geq 1$ relatively prime

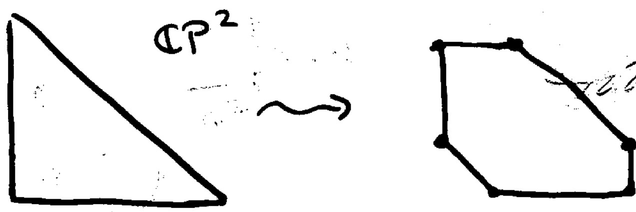
$$\pi_1(B_{n,m}) = \mathbb{Z}/n^2\mathbb{Z}$$

$$H_*(B_{n,m}, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } * = 0 \\ 0 & \text{otherwise} \end{cases}$$

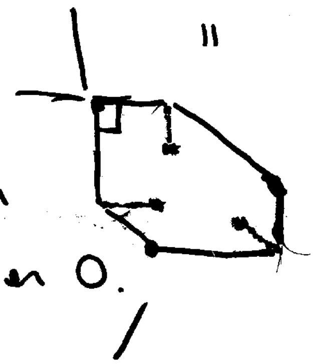
Key: Boundary is ω-**convex**

($\exists$ a $\pitchfork$ vector field X such that $\mathcal{L}_X \omega = \omega$.)

Affine structure on $S^2 - 24$ pts ?
(K3 surface)

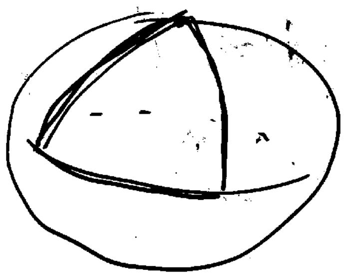


$\parallel$



preimage of bdy is 3 spheres, each with self-intersection 0.

3 nodal fibers



makes up one octant of sphere

$8 \times 3 = 24$!