Some Topological Problems Suggested by Mostow Rigidity.

MOSTOW RIGIDITY : HOMOTOPY EQUIVALENT HYPERBOLIC MANIFOLDS ($n \geq 3$) ARE ISOMETRIC

BOREL CONJECTURE (EQUIVARIANT B.C.)

BAUM-CONNES CONJECTURE

CLASSIFICATION OF HIGH DIMENSIONAL MANIFOLDS WITHIN A GIVEN HOMOTOPY TYPE

NOVIKOV CONJECTURE

(GROMOV-LAWSON (ROSENBERG)) CONJECTURE ON PSC.

HOMOTOPY INVARIANCE (mod Q) OF ATIYAH-PATODI-SINGER INVARIANT

NON-H.E. OF CHEEGER-GROMOV INVARIANTS WHEN π_1 HAS TORSION

NONCOMPACT VERSION:

1. REL ∞ $\Longleftrightarrow$ "STANDARD VERSION"

2. Proper version (usually false)

3. BOUNDED VERSION

STATEMENT

EXAMPLES

PRINCIPLE OF DESCENT
(BOUNDED B.C. $\Longrightarrow$ NOVIKOV CONJECTURE)

IMPLICATIONS

EXPANDERS

COUNTEREXAMPLES

MOSTOW RIGIDITY: Let M^n be a hyperbolic manifold, $n > 2$, then for any N^n hyperbolic and $f : N \to M$ a homotopy equivalence then f is homotopic to a unique isometry.

Cor: $\mathrm{Isom}(M) = \mathrm{Out}(\pi_1 M)$

Cor: Equivariant version of Mostow rigidity.

True for other symmetric spaces, finite volume, for some non-homotopy equivalences (superrigidity) etc...

Borel Conjecture: Let M^n be an aspherical manifold (closed) and $f : N^n \to M$ a homotopy equivalence, then f is homotopic to a homeomorphism.

$n = 3 \Rightarrow$ Poincaré conjecture.

In high dimensions, false smoothly.

True in many cases...

Plausibility theorem is based on examining the difference between

Quadratic form over $\mathbb{R}\Gamma$ $\longrightarrow$ signature

$\downarrow$

$L^2\Gamma$-signature

For $\Gamma = \mathbb{Z}_n$ and the form $\cdot\,(1+g+\dots+g^{n-1})$

$$\operatorname{sign}(\cdot) = 1$$

$$\operatorname{sign}_{\mathbb{Z}_n}(\cdot) = \frac{1}{n}$$

Atiyah's L^2 index theorem asserts that for any operator D on a closed manifold M

$$\operatorname{ind} D = \operatorname{ind}_\Gamma \widetilde{D}$$

[illegible] of [illegible] Conjectures.

① If π is torsion free and BC is true for π then

$$S(M^n) = \{M' \xrightarrow{f} M\}/\text{[illegible]} = H_{n+1}(B\pi, M; \mathbb{L}(e)) \otimes \mathbb{Z}[\tfrac{1}{2}]$$

$$\cong KO_{n+1}(B\pi; M) \otimes \mathbb{Z}[\tfrac{1}{2}]$$

$$\cong \bigoplus_i H_{n+1-4i}(B\pi, M; \mathbb{Q}) \otimes \mathbb{Q}$$

where $\pi = \pi_1 M_{\ldots}$

①b If π has torsion

$$\cong KO^{\pi}_{n+1}(\underline{E\pi}, \tilde{M}) \otimes \mathbb{Z}[\tfrac{1}{2}]$$

(which is somewhat more difficult to write down as ordinary homology)

② ($\Leftarrow$ 2①) (The Novikov Conjecture)

If $f: M' \to M$ is a homotopy equivalence and $\varphi: M \to B\pi$ then

$$\varphi_* f_*(L(M') \cap [M']) = \varphi_*(L(M) \cap [M]) \text{ in } \oplus H_*(B\pi; \mathbb{Q})$$

③ ($\cong$ consequence of (1.5.10 ②))

If $\pi_1 M^{4k+3}$ contains torsion, then $\exists$.
∞'ly many manifolds (simple) homotopy equivalent to M and not homeomorphic to M. $(k>0)$.

Remark: ③ is true unconditionally (Chang - W, '00) The proof is an interplay between [illegible]logy and analysis.

The Baum-Connes Conjecture

Let $\mathbb{C}\Gamma$ act on $\ell^2\Gamma$ by the regular representation; and let $C^*\Gamma$ be the norm completion of this algebra.

The B-C conjecture gives a calculation of $K_0(C^*\Gamma)$ in terms of $K_*^{\Gamma}(\underline{E}\Gamma)$.

BC $\Rightarrow$ N.C.

Because $K(C^*\Gamma)$ is a place where $[\mathrm{sign}\, M]$ is homotopy invariant.

BC $\Rightarrow$ Strong restrictions on which manifolds can have positive scalar curvature.

(Let M be spin, then using

$$\not{D}^*\not{D} = \Delta + \frac{K}{4}$$ (Lichnerowicz formula)

and B.C. one can see that

$$[\not{D}_M] = 0 \in K(C^*\Gamma)$$

so $\varphi_*(\hat{A}(M) \cap [M]) = 0 \in \oplus H_*(B\Gamma; \mathbb{Q})$

etc.

Complete Metrics of P.S.C. on Irreducible symmetric spaces

<u>Defn</u>: Suppose $\Gamma \subset G$ is a lattice, then $\mathbb{Q}\text{-rk}(\Gamma)$ is coarse volume growth of $\Gamma\backslash G/K$

<u>Example</u>: $\mathbb{Q}\text{-rk} = 0 \iff \Gamma$ is cocompact.

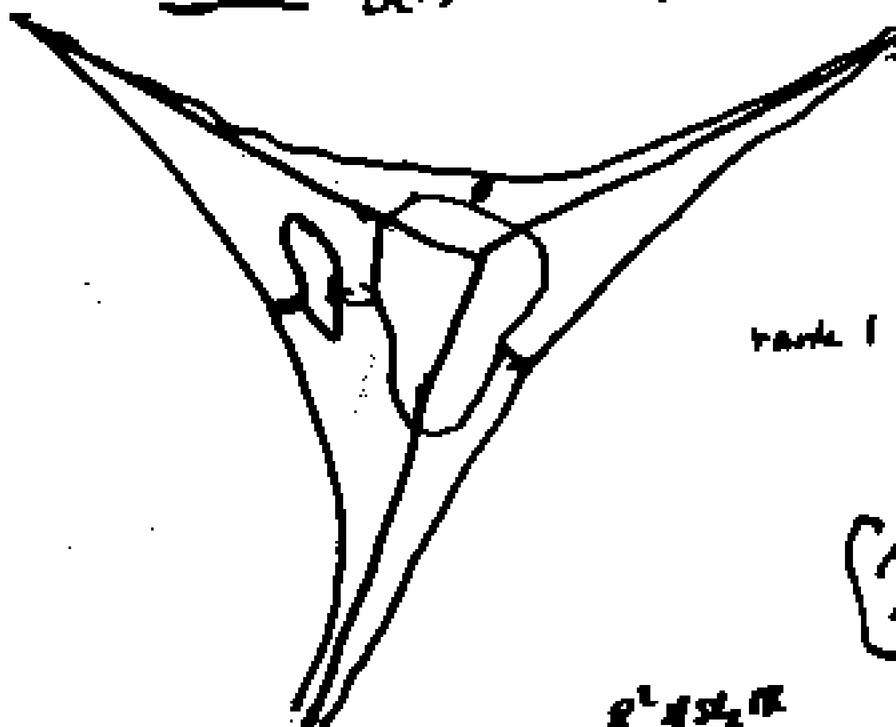
Theorem (Block-W, JDG 1999)
$K\backslash G/\Gamma$ has a complete metric with positive scalar curvature iff $\mathbb{Q}\text{-rk}(\Gamma) > 2$.

Theorem: (Chang, Lubotzky, W) If $\mathbb{Q}\text{-rk}(\Gamma) > 2$
$K\backslash G/\Gamma$ always has a finite sheeted cover which is not properly rigid.

Key philosophical point: $K\backslash G/\Gamma$ is not properly aspherical by Borel-Serre theory.

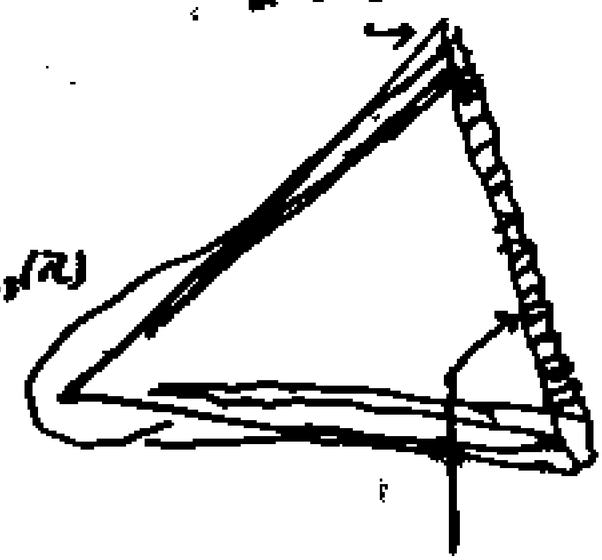
Bounded Asphericality of Arithmetic Manifolds.

Consider: $B(r) \longrightarrow B(f(r))$



rank 1 case

$\begin{pmatrix} A & 0 \\ * \, * & \gamma \end{pmatrix}$



rank 2 case

Special Case: X is UNIFORMLY CONTRACTIBLE if $\exists$ function f such that $\forall x$, $B_x(r) \hookrightarrow B_x(f(r))$ is nullhomotopic.

Coarse analogue of aspherical.

Examples:

1. $\mathbb{R}^n$, ΣP for any polyhedron.

2. $\tilde{X}$ for X a compact aspherical complex.

Conjecture: Uniformly contractible manifolds are boundedly rigid. (even uniformly aspherical)

Conjecture: $K_*^{lf}(X) = K_*$ (Bounded propagation speed operators on X) if X is uniformly contractible.

Some Evidence

Cor. 1. above boils down to the α-approximation theorem of Chapman and Ferry. Here's a closely related theorem of Ferry that gives the flavor.

<u>Theorem</u>: Suppose $n \geq 5$ and M^n is a compact metric topological manifold. Then there is an $\epsilon_M > 0$ such that if

$$f : M \to N^n$$

has $\forall n \in N$ $\operatorname{diam}\{f^{-1}(n)\} < \epsilon$, then f is homotopic to a homeomorphism.

$\epsilon_{\mathbb{R}^n} = 1$.

Theorem (Cheng '99) If $X \approx \mathbb{R}^6/\Gamma$ is a positively scalarly curved manifold, then the diffeomorphism is not a coarse quasi-isometry.

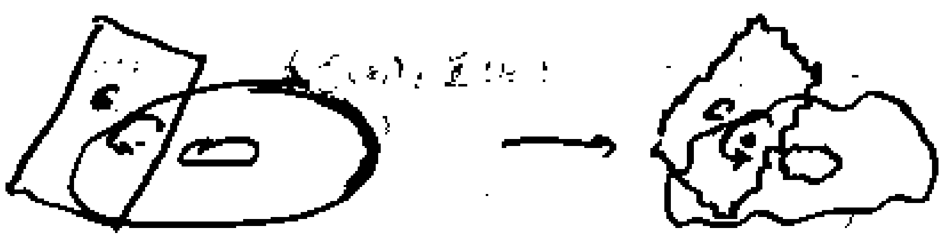
Farrell and Jones proved the "rel ∞" version of BC for $\mathbb{R}^{B}/\Gamma$; the bounded version is true as well (Chang-W).

[Baum-Connes is only known for lattices in rank 1 (Kasparov, Yu...) and for "amenable type groups" (Higson-Kasparov) and cocompact lattices in $SL_3(\mathbb{R})$ (Lafforgue), and a few others.]

PRINCIPLE OF DESCENT.

$$\boxed{BC(|\Gamma|) \Longrightarrow NC(\Gamma)}$$

Proof:



i.e.

$$\begin{array}{ccc} \widetilde{M} \times_\Gamma \widetilde{M} & \longrightarrow & \widetilde{M}' \times_\Gamma \widetilde{M}' \\ \downarrow & & \downarrow \\ M & \longrightarrow & M' \end{array}$$

Q.E.D.

(Ferry-W, Carlsson-Pedersen, Roe, Higson, Kasparov...).

Remark: Descent works in K-theory, L-theory, C^*-algebras; it works equivariantly. It applies to foliations and laminations, etc. With Pedersen

<u>Sample Corollary</u>: If M^n is a symmetric space
any
then $\forall \varepsilon > 0$ there is a section of the map $\mathrm{Diff}(M) \to \mathrm{Homeo}(M)$ within ε of the identity.

Definition. Let Z be a discrete metric space. Z uniformly embeds in Hilbert Space if one has $\Phi : Z \to H$ such that

$$(i) \quad d(\Phi(z_1), \Phi(z_2)) < C\, d(z_1, z_2)$$

and

$$(ii) \quad d(\Phi(z_1), \Phi(z_2)) > f(d(z_1, z_2))$$

Theorem (G. Yu, Inventiones 2000) The Bounded Baum-Connes conjecture holds for X which uniformly embeds in H.

(Based on earlier work of Higson and Kasparov)

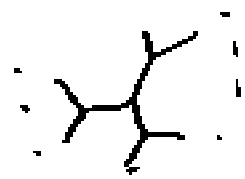
<u>Definition.</u> A sequence of graphs X_i are <u>(k,ε) expanders</u> if

(i) All the X_i's are k-regular.

and (ii) The smallest positive eigenvalue of Δ on X_i are $> \varepsilon$.

$\left(\iff \forall A \subset X_i,\ \#\partial A \geq c(\varepsilon,k) \min(\#A, \#\sim A) \right.$ by Cheeger's inequality$)$

Observation (Gromov) $\coprod X_i$ cannot uniformly embed in H.



<u>Examples:</u> ① Random Graphs

② Quotients of Property T groups

③ $\coprod SL_2(\mathbb{Z}_k)$ (Selberg's theorem on congruence towers)

Theorem (Higson-Lafforgue-Skandalis, Yu etc.)

$\coprod X_i$ expanders has failure of coarse B.C. conjecture.

Consider $P = \bigoplus \frac{1}{\# X_i} \begin{array}{c} X_i \\ \begin{bmatrix} 1 & 1 & 1 & \cdots \\ \vdots & & & \\ \vdots & & & \end{bmatrix} \end{array}$

It is a bounded propagation speed projection not in the image of the assembly map.

Theorem (W-K Whyte) There are disjoint unions of expanders for which the coarse Borel Conjecture is true.

Proof is based on the methods of Farrell and Jones.

Theorem (Dranishnikov - Ferry - W)

For every $n \geq 8$ there are uniformly contractible Riemannian metrics on $\mathbb{R}^n$ which are not boundedly rigid.

Remark: The key to these examples is the analysis of

$$\lim_{R \to \infty} \frac{1}{f(R)} S(R)$$

for an appropriate weighting function f. This limiting object has finite cohomological dimension and infinite dimension. So one doesn't have any coarse failures of Borel conjectures with bounded geometry.