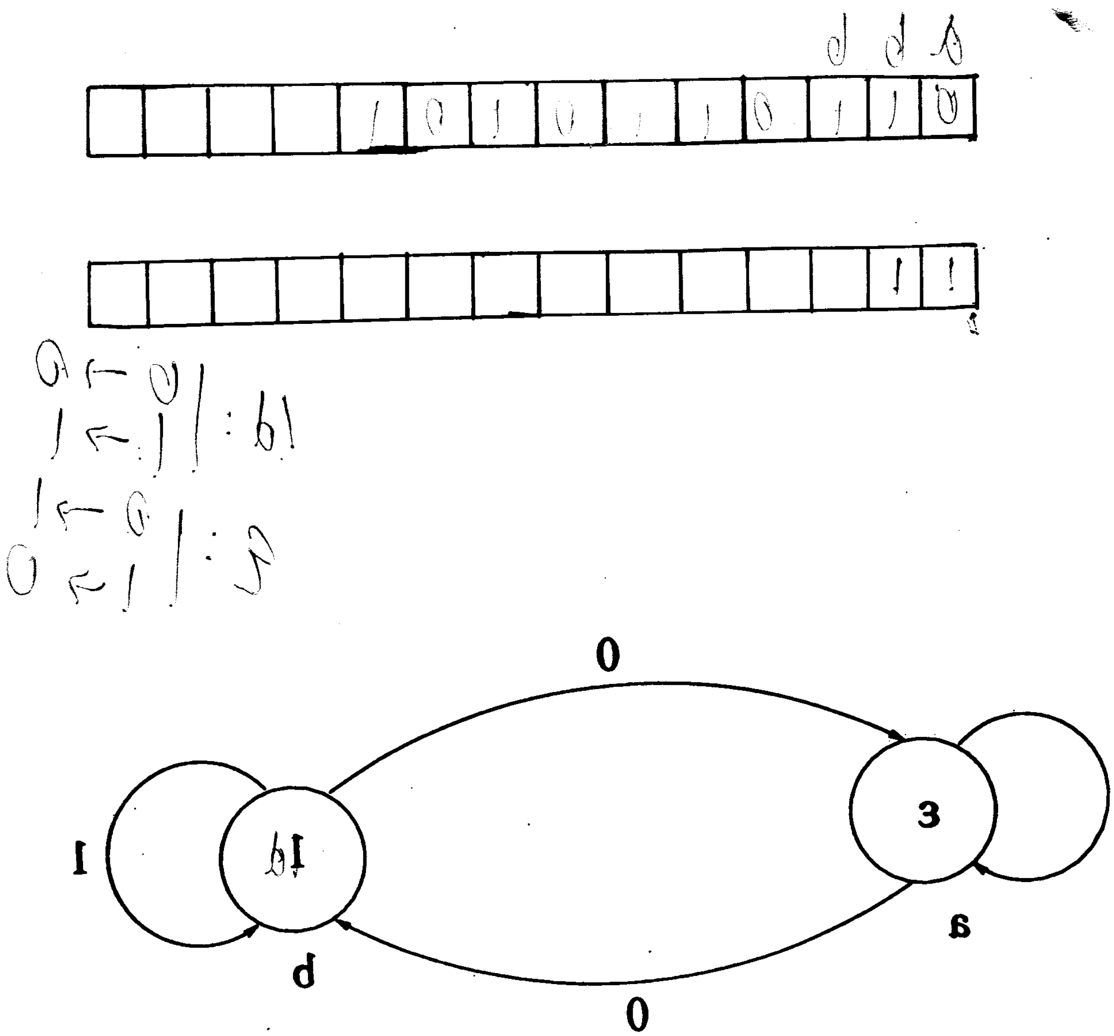
# $L^2$ Betti numbers of closed manifolds

- Automata groups
- Spectral measure
- Atiyah conjecture

joint works with
S. Grigorchuk
&
P. Linnell and T. Schick




A

$$\left(\bigoplus_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}\right) \rtimes \mathbb{Z}$$

the lamplighter group

$$H = \langle a, t \mid a^2 = id,\ [a, t^{-n} a t^n] = id \quad \forall n \in \mathbb{Z} \rangle$$

Kaimanovich, Vershik

1978

# Random walks

$\Gamma$ - discrete group

$S$ - finite, symmetric ($S = S^{-1}$) set of generators

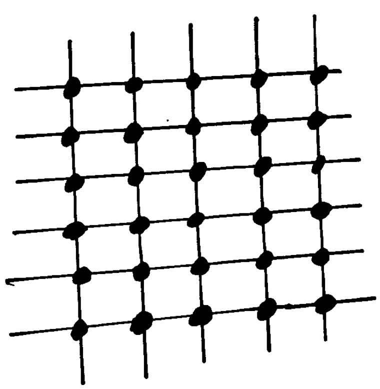
$\mathrm{Cay}(\Gamma, S)$

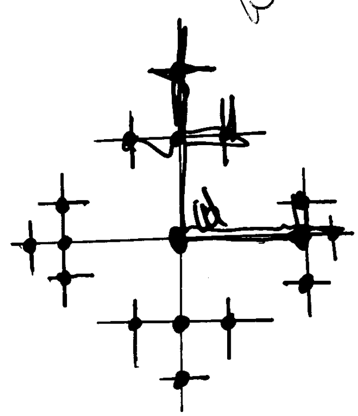
$\mathbb{Z}^2$

$p_n(id, id)$

$\mathbb{F}_2$

Kesten (1959)





$f \in \ell^2(\Gamma)$

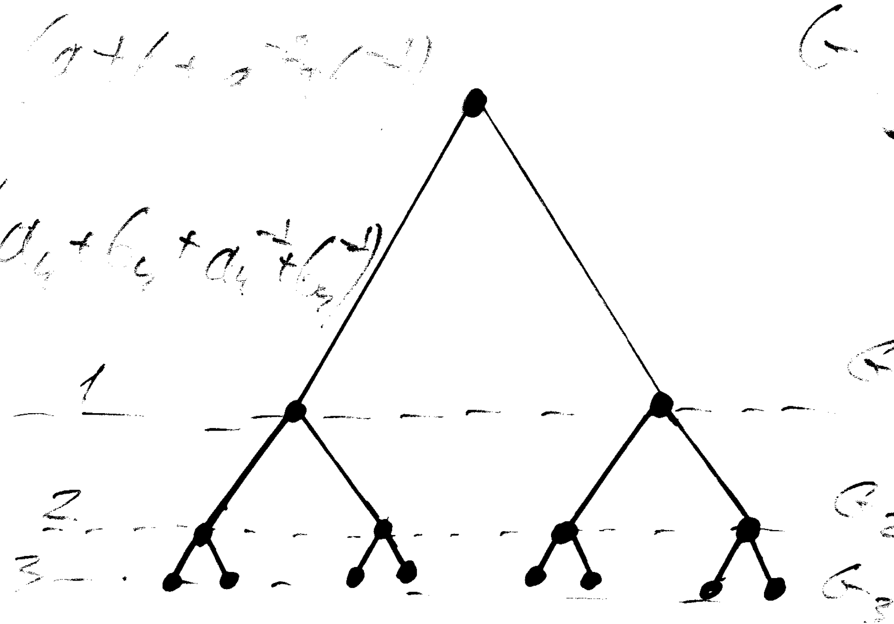
$$M : \ell^2(\Gamma) \to \ell^2(\Gamma)$$

$$Mf(\gamma) = \frac{1}{|S|} \sum_{s \in S} f(\gamma s)$$

$$\|M\|_{\ell^2 \to \ell^2} = \lim_{n \to \infty} \sqrt[2n]{p_{2n}(id, id)}$$

(Kesten, 1959)

$1 \in Sp(M) \Leftrightarrow \Gamma$ est moyennable



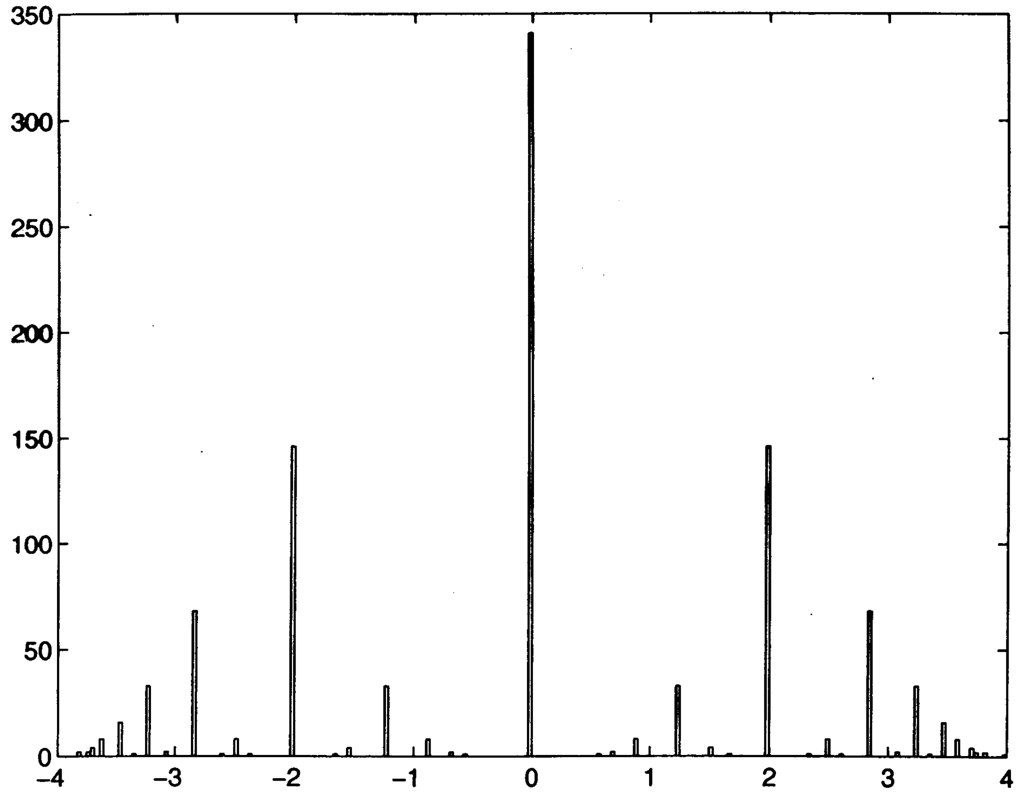

$a \to a_n$ $\quad 2^n \times 2^n$ matrices

$b \to b_n$ $\quad$ with coefficients 0 and 1

$$a_{n+1} = \begin{pmatrix} 0 & a_n \\ b_n & 0 \end{pmatrix}$$

$$b_{n+1} = \begin{pmatrix} a_n & 0 \\ 0 & b_n \end{pmatrix}$$

$$Sp\left(a_n + b_n + a_n^{-1} + b_n^{-1}\right)$$



350
300
250
200
150
100
50
0
−4
−3
−2
−1
0
1
2
3
4

Theorem (Grigorchuk, Ż.) The operator

$A = t + at + t^{-1} + (at)^{-1}$ on $\ell^2(\Gamma)$

has the following eigenvalues $\Gamma \simeq \left(\bigoplus_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}\right) \rtimes \mathbb{Z}$

$$4\cos\left(\frac{p}{q}\pi\right)$$

$q = 2, 3, 4, \dots \qquad p = 1, \dots, q-1$

The dimension of the corresponding eigenspace is equal to

$$H \qquad \dim\left(\operatorname{Ker}\left(A - 4\cos\left(\frac{p}{q}\pi\right)\right)\right) = \frac{1}{2^q - 1}$$

$\langle P_H(\delta_{id}), \delta_{\cancel{id}\,p,q}\rangle = 1.$ $\qquad \underline{\dim(\ker(A)) = \frac{1}{2^2 - 1} = \frac{1}{3}}$

$$(\text{Euler}) \qquad \sum_{q=2}^{\infty} \frac{\varphi(q)}{2^q - 1} = 1.$$

# $L^2$ Betti numbers (Atiyah 1976)

$X$ - finite CW complex

$\pi_1(X) = \Gamma$

$C^*_{(2)}(\tilde{X}) =$ $\Gamma$ equivariant cochains in $\ell^2(\Gamma)$

$d^p_{(2)}$ - coboundary operator

$$H^p_{(2)} = \ker(d^p_{(2)}) \big/ \overline{\operatorname{Im}(d^{p-1}_{(2)})}$$

$$b^p_{(2)} = \dim(H^p_{(2)})$$

$(M, g)$ - closed manifold

$fin^{-1}(\Gamma)$ - the subgroup of $\mathbb{Q}$ generated by the inverses of the orders of finite subgroups of $\Gamma$.

Conjecture: Let $M$ be a closed riemannian manifold s.t. $\pi_1(M) = \Gamma$. Then

$$b^p_{(2)}(\tilde{M}) \in fin^{-1}(\Gamma).$$

The conjecture is true for

- abelian groups (Cohen, 1979)
- elementary amenable groups with the uniform bound on the torsion,
  free groups (Linnell, 1993)

Theorem: Let $\Gamma$ be the group given by the following presentation

$$\Gamma = \langle a, t, s \mid a^2 = id,\ [a, t^{-1}at] = id,\ [t, s] = id,\ s^{-1}as = at^{-1}at \rangle$$

Every finite subgroup of $\Gamma$ has order $2^n$. There exists a closed riemannian manifold $M$ of dimension 7, s.t. $\pi_1(M) = \Gamma$ and

$$b_3^{(2)}(M) = \frac{1}{3}.$$