Hulls, hyperbolicity and rigidity in mapping class groups

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Outline

Mapping class groups

Let S be an oriented surface of finite type.

Definition

$\mathcal{MCG}(S) = Homeo(S)/Homeo_0(S)$

This group is finitely generated and hence has a *word metric*. How do we study the (coarse) geometry of this group?

δ-Hyperbolicity

A group with its word metric (or any geodesic metric space) is called *δ-hyperbolic* (Cannon, Gromov, Rips) if it has *δ-thin triangles.*

That is, for any geodesic triangle $[xy] \cup [yz] \cup [xz]$,

$$[xy] \subset \mathcal{N}_\delta([yz] \cup [xz]).$$

Such groups have a rich and robust geometric theory. (For example, a natural boundary at infinity, stability of quasi-geodesics, a solvable word problem...)

Examples: F_n, cocompact lattices in Isom$(\mathbb{H}^n)$, $SL(2,\mathbb{Z})$,...

Abelian subgroups of $\mathcal{MCG}(S)$

$\mathcal{MCG}(S)$ fails to be hyperbolic, because it contains large-rank abelian subgroups, generated by homeomorphisms supported on disjoint subsurfaces.

Indeed if Δ is a system of disjoint curves in S then $Stab(\Delta)$ has a natural (coarse) product structure, which is quasi-isometrically embedded in $\mathcal{MCG}(S)$

Curve complexes

The attempt to ignore such product regions leads to this definition:

Definition (Harvey)

The *complex of curves* of a surface S is a simplicial complex $\mathcal{C}(S)$ such that:

- Vertices are essential isotopy classes of simple closed curves in S,
- Simplices are $[v_0, \ldots, v_n]$ where the v_i have pairwise disjoint representatives.

(The definition is different in some special cases: annulus, one-holed torus, 4-holed sphere.)

Hyperbolicity regained?

The curve complex gives a *hyperbolic shadow* of $\mathcal{MCG}(S)$:

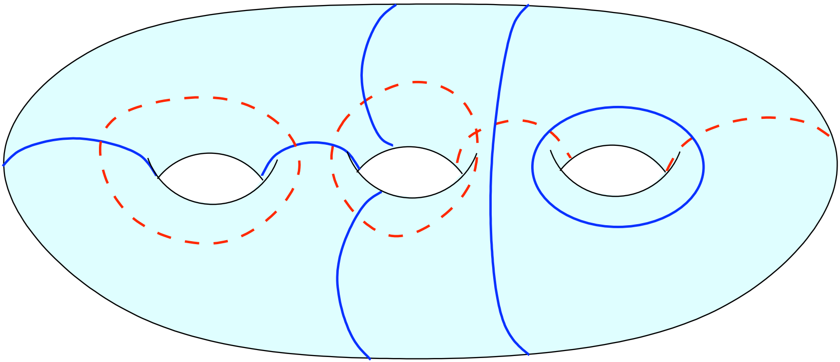
Theorem (Masur-M)

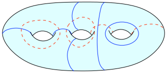
$\mathcal{C}(S)$ is δ-hyperbolic, and (typically) of infinite diameter.

However, a lot of information has been lost. How do we get it back?

A convenient geometric model for $\mathcal{MCG}(S)$

A *marking* on S is a certain standard configuration of curves (considered up to isotopy), filling the surface.





The set of all markings is denoted $\mathcal{M}(S)$, and distance between markings is given by minimal sequences of *elementary moves* between them.
$\mathcal{MCG}(S)$ acts properly and cofinitely on $\mathcal{M}(S)$, hence the two are quasi-isometric.

There is a natural "forgetting" map

$$\pi_S : \mathcal{M}(S) \to \mathcal{C}(S)$$

which is coarse Lipschitz, and has infinite-diameter fibres.

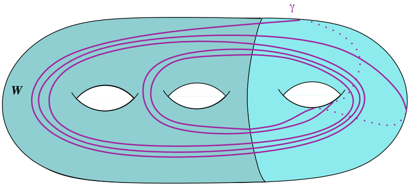
Subsurface projections

The projection to $\mathcal{C}(S)$ can be refined by considering subsurfaces.

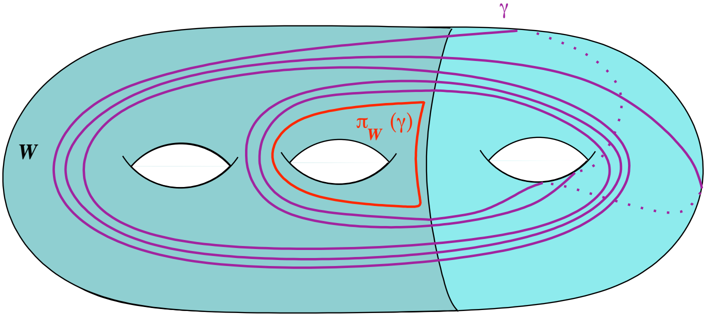
For an essential subsurface $W \subset S$, we define a map

$$\pi_W : \mathcal{M}(S) \to \mathcal{C}(W).$$

Given a curve $\gamma \subset S$, consider its arcs of essential intersection with W. Picking one of these and performing a surgery, we obtain $\pi_W(\gamma)$.



Subsurface projections

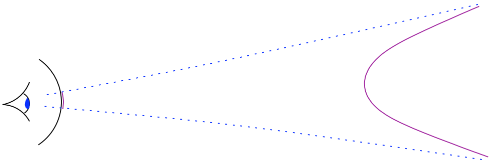


Note that $\pi_W(\gamma)$ is essentially independent of the choices in the construction (up to bounded distance in $\mathcal{C}(W)$)

I am ignoring throughout this talk the case that W is an annulus.

Analogy with visual sphere

$\mathcal{C}(W)$ is contained in the link of $[\partial W]$ in $\mathcal{C}(S)$, which is analogous to a unit sphere of tangent vectors. The analogy between π_W and projection to the unit sphere (in a CAT(0) space) is strengthened by:



Theorem (Bounded Geodesic Projections (Masur-M))

If g is a geodesic in $\mathcal{C}(S)$ which is distance at least 2 from $[\partial W]$ then

$$\mathrm{diam}(\pi_W(g)) \leq B$$

where B is uniform.

The global projection map

We can combine all these into one large map:

$$\Pi : \mathcal{M}(S) \to \prod_{W \subseteq S} \mathcal{C}(W)$$

which is coarsely distance-preserving in this sense:

Theorem (Masur-M)

For $\mu, \nu \in \mathcal{M}(S)$, and uniform K,

$$d(\mu,\nu) \sim \sum_{W \subseteq S} [[d_W(\mu,\nu)]]_K$$

(Here $[[x]]_K$ is 0 if $x < K$, and x if $x \geq K$. All estimates are up to uniform multiplicative and additive error)

The image of Π

So, Π embeds $\mathcal{M}(S)$ into an infinite product of δ-hyperbolic spaces. What is this good for?

First of all, What is the image of Π like?

Behrstock proved that there are certain constraints on the image of Π. (Simplified proof due to Leininger)

Pairs of subsurfaces

For any pair U, V of subsurfaces, we say that $U \pitchfork V$ if they intersect essentially but are not nested. (We say that they *overlap*) We can ask what is the image of

$$\pi_U \times \pi_V : \mathcal{M}(S) \to \mathcal{C}(U) \times \mathcal{C}(V)$$

and the answer depends on the three possible configurations:

- $U \cap V = \emptyset$
- $U \pitchfork V$
- $U \subset V$

Disjoint Case

If $U \cap V = \emptyset$ then there are no constraints: the image is all of $\mathcal{C}(U) \times \mathcal{C}(V)$.

Simply build a marking starting with $\partial U \cup \partial V$ and add whatever you want.

Overlapping Case

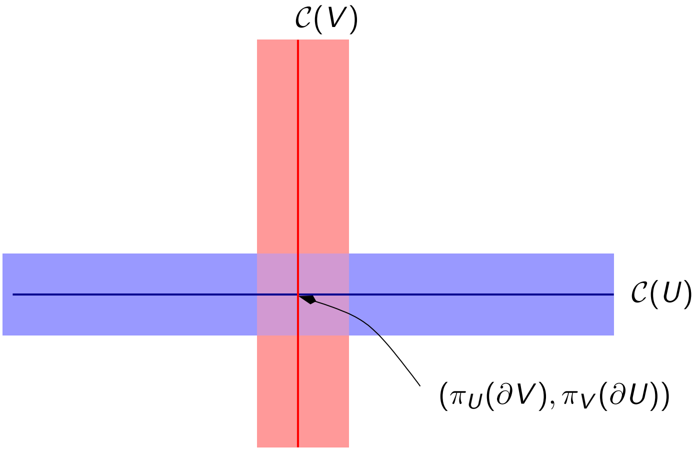
First consistency inequality

If $U \pitchfork V$ then

$$\min\left(d_U(\partial V, \mu), d_V(\partial U, \mu)\right) \leq B$$

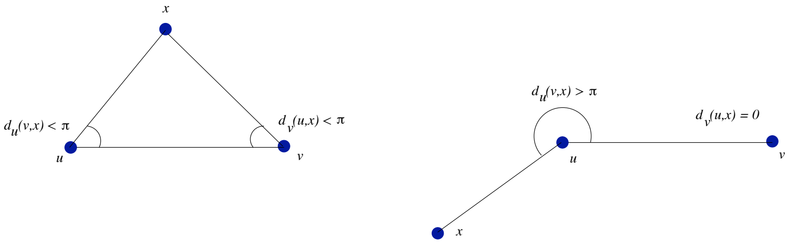
for any $\mu \in \mathcal{M}(S)$ where B depends only on S.

$$\min\left(d_U(\partial V,\mu), d_V(\partial U,\mu)\right) \le B$$



The CAT(0) analogy

In a CAT(0) simplicial complex, angles can be any positive number, but an embedded triangle has angles less than π. Hence:



Nested Case

Second consistency inequality

If $U \subsetneq V$ and $d_V(\partial U, \mu) > 1$, then

$$d_U(\pi_U(\mu), \pi_U(\pi_V(\mu))) \leq B$$

This is essentially the fact that, when both projections are defined, they compose naturally.

Consistency Theorem

Behrstock's inequalities turn out to characterize the image completely (in a coarse sense):

Theorem (Behrstock-Kleiner-M-Mosher)

A point $(x_W) \in \prod_W \mathcal{C}(W)$ is close to the image of Π if and only if it coarsely satisfies the consistency inequalities for all subsurface pairs.

More precisely, we coarsify the inequalities as follows: There exist constants m_1, m_2 such that, for all $U, V \subseteq S$,

1. If $U \pitchfork V$ then

$$\min\left(d_U(\partial V, x_U), d_V(\partial U, x_V)\right) \leq m_1.$$

2. If $U \subsetneq V$, and $d_V(\partial U, x_V) \geq m_2$, then

$$d_U(x_U, x_V) \leq m_1.$$

The theorem says that, given B, there exist m_1, m_2 such that, if (x_W) satisfies (1) and (2) with these constants, then there exists $\mu \in \mathcal{M}(S)$ such that for all W

$$d_W(\mu, x_W) \leq B.$$

Applications of the consistency theorem

The Consistency Theorem allows us to make constructions in $\mathcal{M}(S)$ by first doing something in the hyperbolic factors $\mathcal{C}(W)$, and then checking that the consistency conditions hold.

A particularly useful construction is a certain notion of *hulls* of finite sets of points in $\mathcal{M}(S)$.

Hulls in hyperbolic spaces

Let X be δ-hyperbolic and $A \subset X$ a finite set.

Definition (hyperbolic hull)

Let $\mathrm{hull}_X(A)$ be the union of all geodesic segments $[a, b]$ where $a, b \in A$.

This is well-behaved in the following senses:

Lemma

1. $\mathrm{diam}(\mathrm{hull}_X(A)) \leq K\mathrm{diam}(A)$ *where* K *depends on* δ.
2. *The nearest point projection* $\varphi_A : X \to \mathrm{hull}_X(A)$ *is coarse-lipschitz as a function of both* A *and* $x \in X$.
3. $\mathrm{hull}_X(A)$ *is coarsely equivalent to a tree.*

(Constants depend only on δ *and* $\#A$*)*

Hulls in $\mathcal{M}(S)$

If $A \subset \mathcal{M}(S)$ and $W \subset S$ let $\text{hull}_W(A)$ be the hyperbolic hull of $\pi_W(A)$ in $\mathcal{C}(W)$.

Definition

Let $A \subset \mathcal{M}(S)$ be a finite set, and $\epsilon > 0$.

$$\Sigma_\epsilon(A) \equiv \{\mu \in \mathcal{M}(S) : \forall W \subset S, \pi_W(\mu) \in \mathcal{N}_\epsilon(\text{hull}_W(A))\}$$

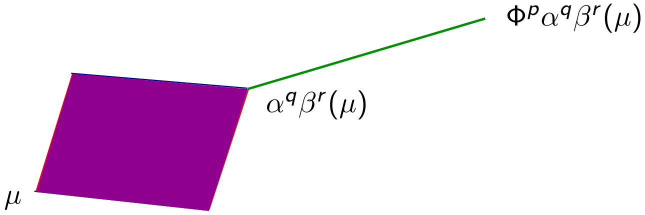
(Usually ϵ will need to be larger than some threshold)

Examples of hulls

Let $\mu \in \mathcal{M}(S)$, $X, Y \subset S$ disjoint subsurfaces, and $\alpha, \beta \in \mathcal{MCG}(S)$ supported in X and Y, respectively (in particular they commute).
Suppose $\alpha|_X$ and $\beta|_Y$ are Pseudo-Anosov. Let Φ be Pseudo-Anosov in S. The hull

$$\Sigma_\epsilon(\mu, \Phi^p \alpha^q \beta^r(\mu))$$

is controlled by its projections into $\mathcal{C}(S)$, $\mathcal{C}(X)$ and $\mathcal{C}(Y)$.



Hull Theorem

Hulls have controlled geometry:

Theorem (Hull projection theorem)

1. *There exists for all finite A a map $\varphi_A : \mathcal{M}(S) \to \Sigma_\epsilon(A)$ such that*
   - $\varphi_A|_{\Sigma_\epsilon(A)} = id$
   - $(A, x) \mapsto \varphi_A(x)$ *is uniformly coarse-Lipschitz*
2. $\mathrm{diam}(\Sigma_\epsilon(A)) \leq K\mathrm{diam}(A)$
3. $\Sigma_\epsilon(A)$ *is coarsely contractible.*

All constants depend only on the cardinality of A

Proof of Hull Theorem, 1

It is not hard to see that (1) $\implies$ (2) and (3). Idea of proof of (1):

- For hyperbolic hulls, (1) is just the lemma on nearest-point retractions.
- So for $x \in \mathcal{M}(S)$ and $W \subseteq S$, define y_W to be the nearest point to $\pi_W(x)$ on $\text{hull}_W(A)$.
- *Claim: the tuple (y_W) satisfies the consistency inequalities.*
- Hence by Consistency Theorem there is $\mu \in \mathcal{M}(S)$ such that $\pi_W(\mu) \sim y_W$. Define $\varphi_A(x) \equiv \mu$. Properties now follow from hyperbolic case.

Proof of Hull Theorem, 2 (Consistency)

Consistency condition 1: Given $U, V \subset S$ with $U \pitchfork V$, prove that

$$\min(d_U(\partial V, y_U), d_V(\partial U, y_V)) \text{ is small.}$$

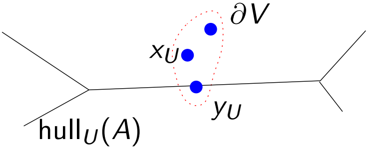
By consistency for x we can assume w.l.o.g.

$$d_U(\partial V, x) \text{ small.}$$

Looking in $\mathcal{C}(U)$, there are two cases, depending on the position of $x_U = \pi_U(x)$.

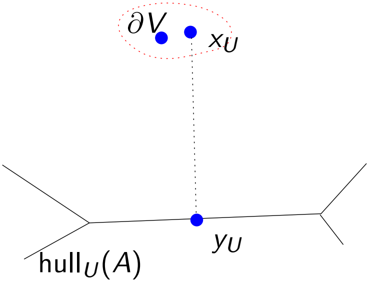
Proof of Hull Theorem, 3 (Consistency)

Case 1: x_U close to $\mathrm{hull}_U(A)$:



So $d_U(\partial V, y_U)$ is small.

Case 2: x_U far from $\mathrm{hull}_U(A)$.



For any $a \in A$, $d_U(\partial V, a)$ is large, so $d_V(\partial U, a)$ is small. Hence $\mathrm{hull}_V(A)$ is near $\pi_V(\partial U)$. Since $y_V \in \mathrm{hull}_V(A)$, $d_V(\partial U, y_V)$ is small.

Proof of Hull Theorem, 4 (Consistency)

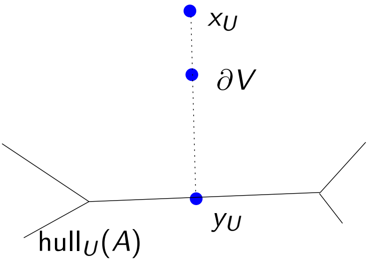
Consistency condition 2: Given $V \subset U \subseteq S$, prove that

$$d_U(\partial V, y_U) \text{ large } \implies d_V(y_V, y_U) \text{ small.}$$

Again there are two cases, now depending on the position of ∂V relative to the $\mathcal{C}(U)$-geodesic $[x_U, y_U]$.

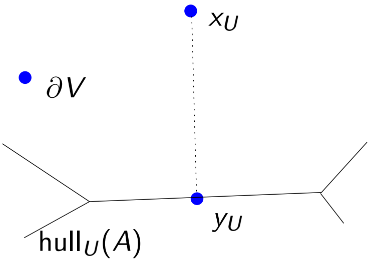
Proof of Hull Theorem, 5 (Consistency)

Case 1: ∂V close to $[x_U, y_U]$.



Geodesic Projection Thm $\Longrightarrow$ $\mathrm{diam}(\pi_V(A)) \sim \mathrm{diam}(\pi_V(\pi_U(A)))$ must be small, and hull_V contains both y_V and $\pi_V(y_U)$. Hence $d_V(y_V, y_U)$ is small.

Case 2: ∂V is far from $[x_U, y_U]$.



Geodesic Projection Thm $\Longrightarrow$ $d_V(x_U, y_U)$ small, so $d_V(x_V, y_U)$ is small. Since $\pi_V(y_U) \in \mathrm{hull}_V(A)$ and y_V is the closest point to x_V, $d_V(y_V, y_U)$ is small.

Other properties of hulls:

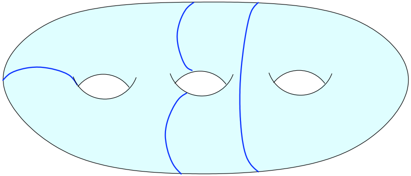
- Polynomial growth:

$$\#\Sigma_\epsilon(A) \leq (\mathrm{diam}(A))^{\xi(S)}$$

where $\xi(S) = 3g - 3 + p$

- Cubes:

Let $Q(\Delta)$ be the set of markings containing a curve system Δ, and suppose components $R \subset S \setminus \Delta$ are of complexity $\xi(R) \leq 1$. Then if $x, y \in Q(\Delta)$ then $\Sigma_\epsilon(x, y)$ is coarsely isometric to a cube of dimension $\leq \xi(S)$.



Structure of Asymptotic cone

The asymptotic cone $Cone(\mathcal{M}(S))$ is, informally, a rescaling limit

$$\lim_{n\to\infty}(\mathcal{M}(S), \frac{1}{s_n}d_{\mathcal{M}})$$

where $s_n \to \infty$, and the limit is taken by means of an ultrafilter. Coarse Lipschitz maps become Lipschitz in the cone, and coarse contractibility of hulls becomes actual contractibility.

Hence, hulls give us a tool for "straightening" singular chains in the cone, and hence controlling topological properties such as homological dimension (cf. Hamenstadt)

More precisely, the hull projection theorem in the limit allows us to define hulls $\Sigma(A)$ in $Cone(\mathcal{M}(S))$, such that

Theorem (Asymptotic hulls)

For a finite set $A \subset Cone(\mathcal{M}(S))$,

- $\Sigma(A)$ *admits a Lipschitz retraction* $Cone(\mathcal{M}(S)) \to \Sigma(A)$,
- $\mathrm{diam}(\Sigma(A)) \leq K\mathrm{diam}(A)$
- $\Sigma(A)$ *is contractible.*

Straightening chains

A k-chain $\eta \in C_k(Cone(\mathcal{M}(S))$ is *straight* if for each simplex δ, $\eta(\delta) \subset \Sigma(\delta^0)$.
The asymptotic hull theorem implies that any chain can be approximated by straight chains. One can use this to obtain another proof of

Theorem (Homological dimension (Hamenstadt))

If $W \subset U$ are open subsets of $Cone(\mathcal{M}(S))$ then

$$H_k(U, W) = 0$$

for $k > \xi(S)$.

We can use this to show (following Kleiner-Leeb) that

Lemma

If $U \subset Cone(\mathcal{M}(S))$ is a top-dimensional manifold, then for $p \in U$

$$H_n(U, U-p) \to H_n(Cone(\mathcal{M}(S)), Cone(\mathcal{M}(S)) - p)$$

is injective (where $n = \xi(S)$).

It follows that, locally, U is *contained* in its straight approximations.

Cubes and the finiteness theorem

A finer analysis then shows, recalling the notion of *cubes* associated to a curve system Δ,

Theorem (Behrstock-Kleiner-M-Mosher)

A manifold of maximal dimension in $Cone(\mathcal{M}(S))$ *is locally contained in a finite union of cubes*

Preservation of Dehn twist flats

The finiteness theorem controls homeomorphisms of $Cone(\mathcal{M}(S))$. We consider maximal Dehn twist flats, which in the cone become topological flats. The image of such a flat is locally a finite union of cubes, and this can be analyzed combinatorially, yielding

Theorem

Any homeomorphism of $Cone(\mathcal{M}(S))$ must permute the set of Dehn twist flats.

Rigidity

Permutations of Dehn twist flats are controlled by Ivanov's theorem on automorphisms of $\mathcal{C}(S)$, yielding these quasi-isometric rigidity results for $\mathcal{MCG}(S)$:

Theorem (Behrstock-Kleiner-M-Mosher)

Any quasi-isometry of $\mathcal{MCG}(S)$ is within finite distance of left-multiplication by an element of the group. (There is a small list of standard exceptions)

Theorem (Hamenstadt, BKMM)

Any group quasi-isometric to $\mathcal{MCG}(S)$ differs from it only by restrictions to finite-index subgroups and quotients by finite subgroups.

Other applications

- Rapid Decay property for $\mathcal{MCG}(S)$ [Behrstock-M] (uses the polynomial growth of hulls)
- Tree-graded structure of $\mathcal{MCG}(S)$ [BKMM] (using hulls and subsurface projections to control separation properties in the cone)
- Constraints on subgroups with property (T) [Behrstock-Drutu-Sapir,Groves]