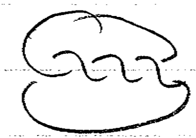
Khovanov Homology (Categorification of the Jones polyn)

I. The Jones polyn. Inv. of links in $\mathbb{R}^3$

Kauffman bracket: define for a knot diagram

$\langle L \rangle$ for a diagram L.



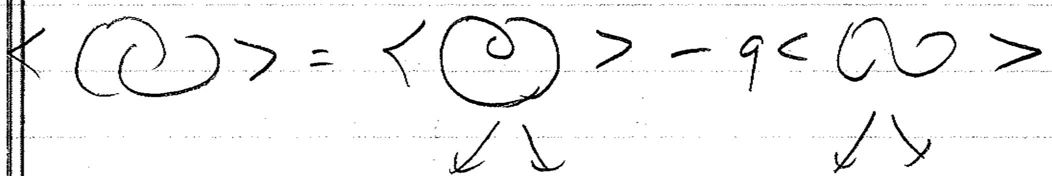
Prop. 1) $\langle O \rangle = q + q^{-1}$

2) $\langle L_1 \sqcup L_2 \rangle = \langle L_1 \rangle \cdot \langle L_2 \rangle$

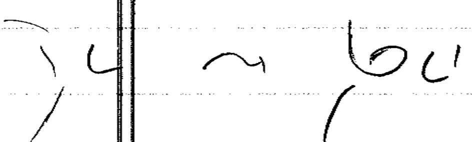
3) $\langle \times \rangle = \langle \asymp \rangle - q \langle)(\rangle$

From this can compute $\langle\ \rangle$ for any diagram

eg:



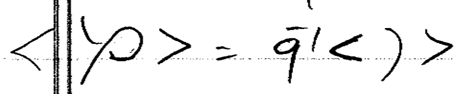
Rk: Kauffman bracket is not a link invariant.



$\langle L' \rangle = \langle \ \rangle - q \langle)\, O \rangle = \langle \ \rangle - q \langle) \rangle \cdot \langle O \rangle$

$= \langle L \rangle - q \langle L \rangle (q + q^{-1})$

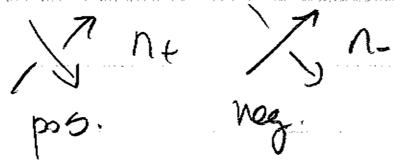
$= -q^2 \langle L \rangle$

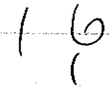
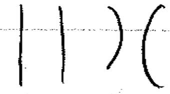
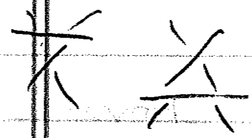


To get an invariant, consider

$\hat{J}(L) = (-1)^{n_-} q^{n_+ - 2n_-} \langle L \rangle$ (unnormalized) Jones polyn.

$\hat{J}(L)$ = link invariant.



Reidemeister moves R1  R2  R3 

$J(L) = (q+q^{-1})^{-1} \hat{J}(L)$

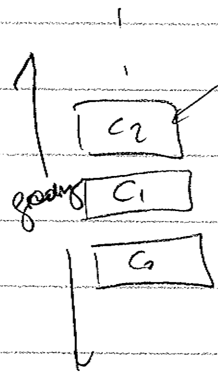
why categorification?

numerical inv. $\overset{?}{\rightsquigarrow}$ Homology theory whose Euler char. is that num. inv.

ex. Top. space Euler char.
simp. space

Hom theory (sing. homology) whose Euler char is χ ! (gives an invariant)
simp. hom.

Hom. better than χ because 1. more info.
2. maps $f: M \to N$
$f_* : H_*(M) \to H_*(N)$



chain gps generated by simplices of corresp. dim.

$\chi = |C_0| - |C_1| + |C_2| \cdots$

categorifications

Jones ⇝ Khovanov

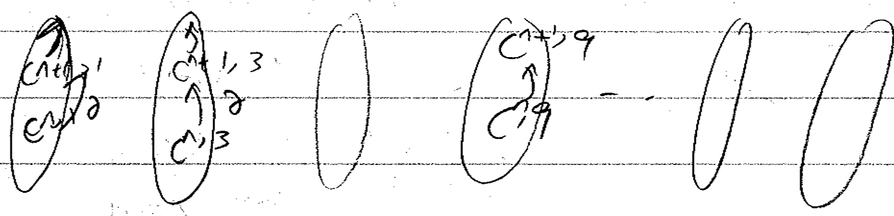
Alexander ⇝ HFK

HomFlyPt ⇝ ... there's sth also categorize this

Categorification of a polyn.

trefoil: $1 \cdot q + q^3 + q^5 - q^9$

want chain cx for each



want: bigraded complex

$$CKH = \bigoplus_{n,J} C^{n,J}$$

n = hom. grading

J = quantum grading

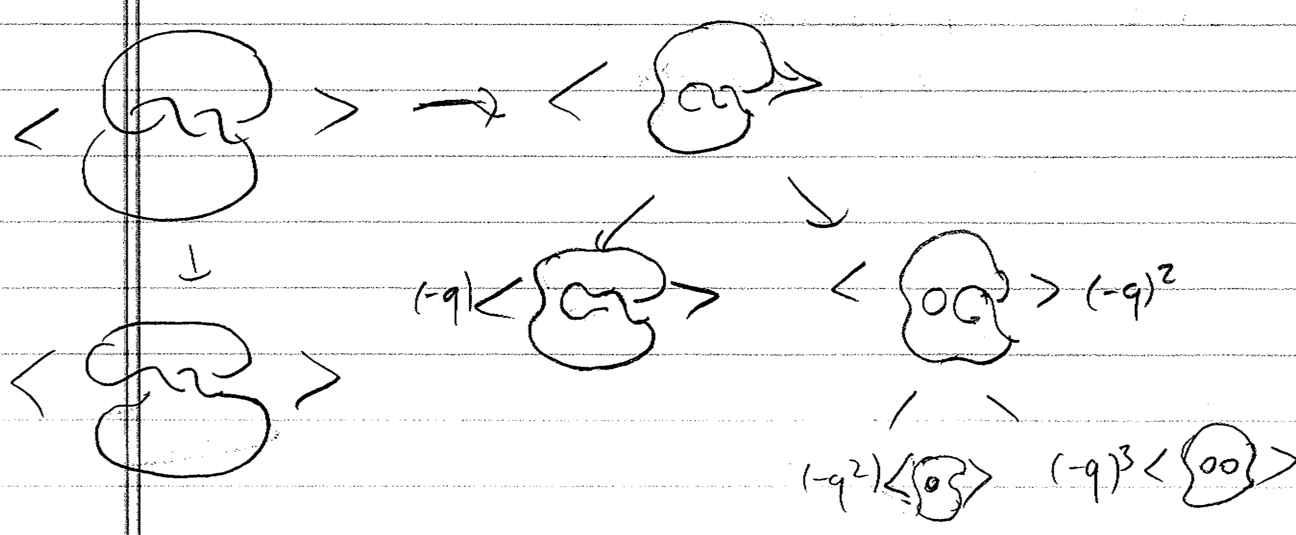
$$C^{n,J} \xrightarrow{\partial} C^{n+1,J}$$

∂ preserves quantum grading, raises homological gr by 1.

$\sum_n (-1)^n \operatorname{rk} H^{n,J}$ = coeff at q^J in the Jones polyn.

$$\hat{J}(q) = \sum_J \Big(\sum_n (-1)^n \operatorname{rk} H^{n,J} \Big) q^J$$

construct CKH, ∂ Motivation back to computing Jones.



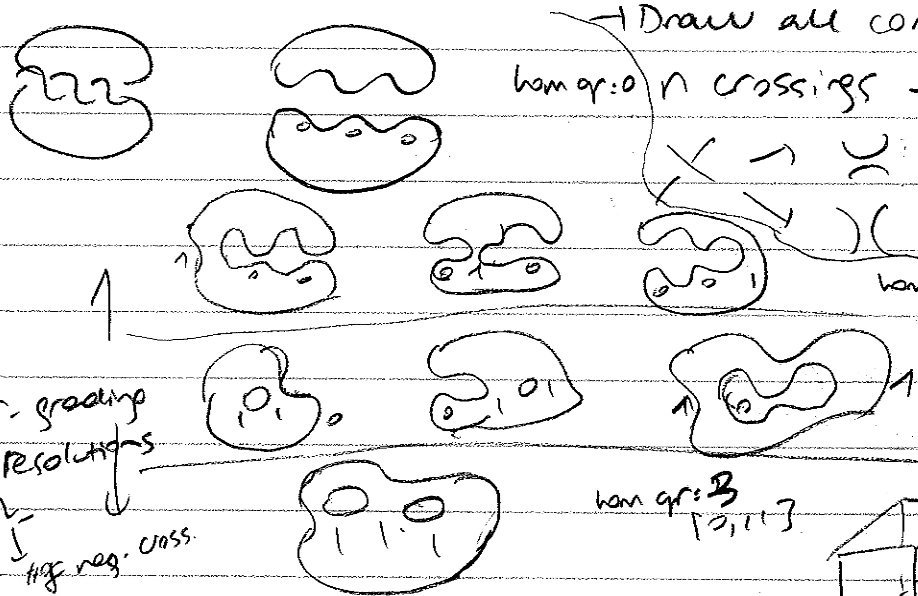
Each "complete resolution" contributes into < >

$$(-q)^{\#(\times \to)(\,)}\,(q+q^{-1})^{\#(\text{comp. in res})}$$

Define CKh Khovanov complex from link diagram

→ Draw all complete resolutions

$\mathbb{Z}_2$



hom gr: 0

n crossings → 2^n complete res.

$\times \to$ ≍ 0-res

$\times \to$)(1-res.

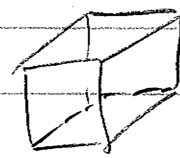
contributes $(q+q^{-1})^{\#\text{comp.}}$

hom. gr = 1

hom gr = 2

hom gr: 3 [0,1,1]

hom. grading = # 1-resolutions − n_- (# of neg. cross.)



'cube of resol.'

$$CKh = \bigoplus_{\text{all resolutions}} CKh(\,\circ_\circ\circ\,) \quad \text{bigraded.}$$

Define $CKh(\,\circ\circ\circ\,)\ V^{\otimes m}$

Pick V 2-dim vsp. w/ basis v_+, v_-

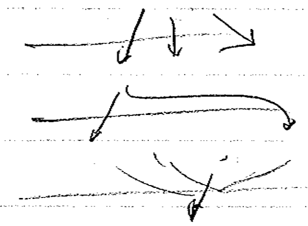
$+1 \leftarrow v_+$ quantum grading
$-1 \leftarrow v_-$

$v_+ \otimes v_- \otimes v_+ \otimes \cdots$ q grading on genera.

$$= \# v_+ - \# v_- + \# 1\text{-res.} + n_+ - 2n_-$$

graded Euler char. $q\text{-}\chi(V) = (q^{+1} - q^{-1})^M$

Define ∂ on each CKh(…) separately.

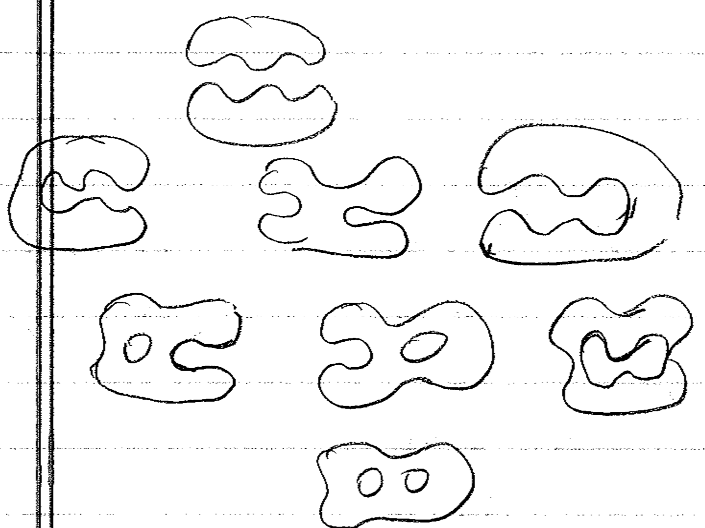


∂: sum of ∂_i

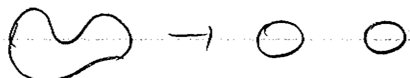
∂_i: CKh(…) $\longrightarrow$ CKh(   )(   )

what are the ∂_i's?

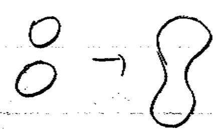
for the eg. before.



w/ every resolution two possible scenarios:

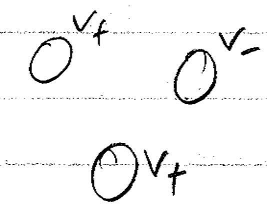
A. 

a circle splits

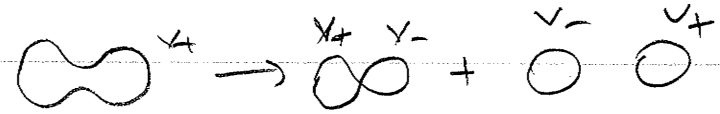
B. 

two circles merge

other components unaffected.

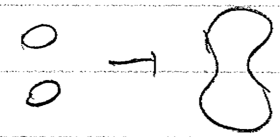


$V_+ \otimes V_- \otimes V_+ \otimes \ldots$

A. 

$v_+ \to v_+ \otimes v_- + v_- \otimes v_+$

$v_- \to v_- \otimes v_-$

B. 

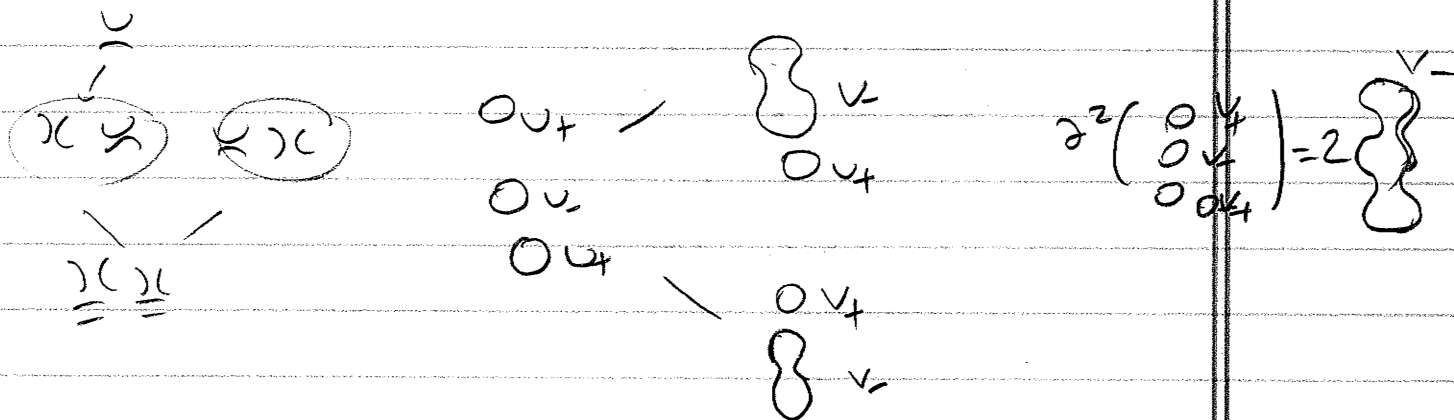
$v_+ \otimes v_+ \to v_+$

$v_+ \otimes v_- \to v_-$

$v_- \otimes v_- \to 0$

This is a chain complex

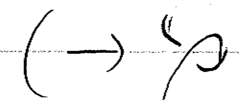
chain: $\partial^2 = 0$ $\mathbb{Z}_2$

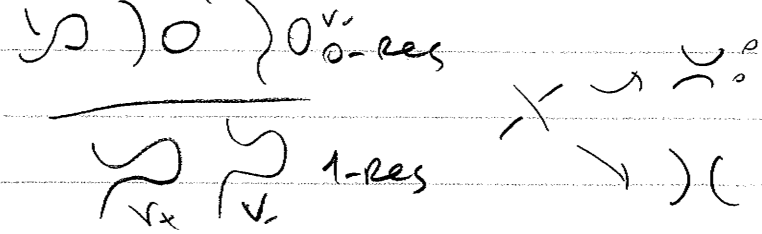


link diag ⇝ chain complex

Thm (Kh) Homology of (CKh, ∂) is a link invariant (ie. independent of diagram)

check: Reidemeister moves give quasi-isom. of chain complex.

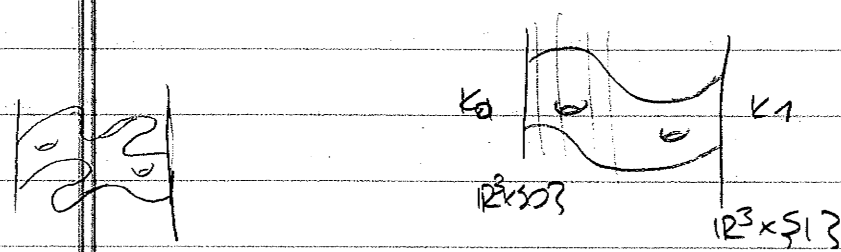
check RI 



$Kh = H_*(Kh, \partial)$

TQFT properties

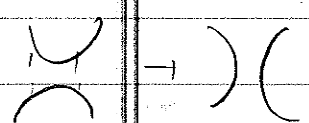
cobordism between knots, S embedded surface in $\mathbb{R}^3 \times [0,1]$



$K_i = S \cap [\mathbb{R}^3 \times \{i\}]$
$i = 0, 1$

claim: S induces a map between Kh.

$f_S : Kh(K_0) \to Kh(K_1)$ f_S shifts quantum degree by $-\chi(S)$



nothing $\to O^{v_+}$
$v_+ O \to$ nothing

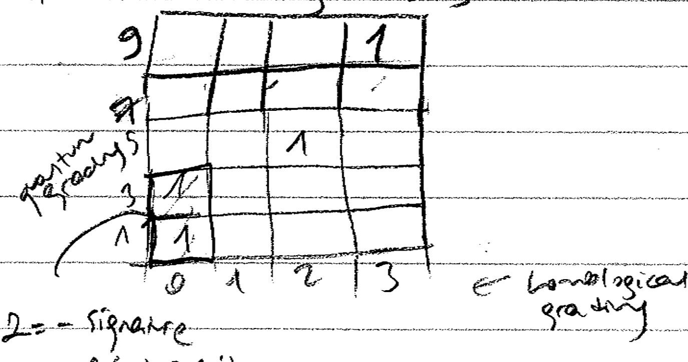
$v_- \to 1$
$v_+ \to 0$

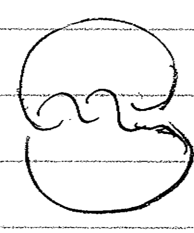
decompose to Morse Moves

For each change in K_t, associate a map
Reidemeister moves: have quasi-isom on chain cpx.

if there is a cobordism between knots there is a well-defined map between Kh hom.

Computing Kh:
everything is combinatorial even the chain complex is huge
Kh known for Alternating knots





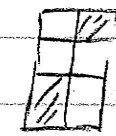
Jones poly.

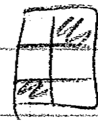
$q + q^3 + q^7 - q^9$

$j-2n=-\sigma\pm 1$

Thm (Lee) for alternating knots Kh supported on 2 diag.

$j-2n=-\sigma\pm 1$



. apart from two rank 1 homologies in hom gr=0, quant gr= $\sigma\pm 1$
has a matching pattern  take it apart the rest breaks into pairs.

s invariant:

Lee: $\exists$ spectral seq. (for knots) from Kh to $\mathbb{Q}\oplus\mathbb{Q}$
$s-1$ $s+1$

Rasmussen

Is important using this

. Milnor conj. is proved combinatorically.

. slice genus can be computed for torus knots.