Geometry of the mapping class group

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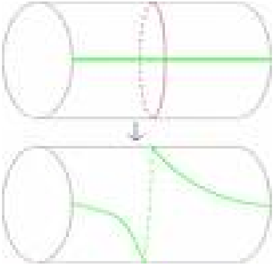
Thm (Dehn, Lickorich): $\mathrm{Mod}(S)$ is generated by finitely many Dehn twists.

W. Lickorich 1964: $3g - 1$ explicitly chosen Dehn twists suffice
S. Humphries 1977: $2g + 1$ Dehn twist suffice (optimal)

Def: Dehn twist



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9. $\mathrm{Mod}(S)$ satisfies the Novikov conjecture.

Def: A finite symmetric set of generators of $\mathrm{Mod}(S)$ defines a Cayley graph for $\mathrm{Mod}(S)$.

Cayley-graph $=$ locally compact metric $\mathrm{Mod}(S)$-graph.

Def: An L-quasi-isometric embedding ($L \geq 1$) of a metric spaces (X, d) into a metric space (Y, d) is a map $F : X \to Y$ s.th.

$$d(x, y)/L - L \leq d(Fx, Fy) \leq Ld(x, y) + L \, \forall x, y \in X.$$

F is a L-quasi-isometry if moreover

$$\forall y \in Y \text{ there is } x \in X : d(Fx, y) \leq L.$$

Fact: Any two Cayley graphs are quasi-isometric.

Basic questions:

1. How does the geometry of $\mathrm{Mod}(S)$ (i.e the geometry of its Cayley graph) look like?
2. What does the geometry tell us about the group structure?
3. What properties does $\mathrm{Mod}(S)$ have? Are there nice spaces on which $\mathrm{Mod}(S)$ acts?
4. Is $\mathrm{Mod}(S)$ similar to a familiar group, like a lattice in a semi-simple Lie group of non-compact type?

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Basic idea: Try to understand the group $\mathrm{Mod}(S)$ by a nice space on which it acts isometrically, properly and cocompactly.

Geometric properties of groups: Semi-hyperbolicity

Let Γ be a finitely generated group.
A *bicombing* of Γ is the assignment of a "path" $\rho_{x,y} : [0,1] \to \Gamma$ connecting $\rho(0) = x$ to $\rho(1) = y$.
The bicombing is *quasi-convex* if there exists $L \geq 1$ s.th.

$$d(\rho_{x,y}(s), \rho_{z,u}(s)) \leq L(d(x,z) + d(y,u)).$$

The bicombing is *invariant* if $\rho_{gx,gy} = g\rho_{x,y} \, \forall g, x, y \in \Gamma$.

The group is *semi-hyperbolic* if it admits an invariant quasi-convex bicombing.

Examples:

A group acting properly, isometrically and cocompactly on a $\mathrm{CAT}(0)$-space is semi-hyperbolic.

A word hyperbolic group is semi-hyperbolic.

A *regular language* over a finite alphabet A is given by a machine which

> takes as input any *word* in A
>
> either accepts the word or stops and rejects the word

with a procedure as follows:

There are finitely many *states* the machine can take.
The machine starts with a *beginning state.*
At each step, the machine reads a letter from A and, based on that letter and the state it is in, it changes to a new state.
There is a *rejection state* which forces the machine to stop.

Biautomatic structures for groups:

Let Γ be a finitely generated group. A biautomatic structure for Γ consists of

1. A finite alphabet A and a map $\pi : A \to \Gamma$ so that $\pi(A)$ generates Γ.
2. There is an inversion $\iota : A \to A$ with $\pi(\iota a) = \pi(a)^{-1}$ for all $a \in A$.
3. A regular language L with alphabet A such that $\pi(L) = \Gamma$.
4. Let $x, y \in A, w \in L$, $w' \in L$ so that $\pi(xwy) = \pi(w')$. Then the paths in Γ defined by xwy, w' are uniform *fellow travellers*.

Means: The regular language defines a "bicombing" of the group.

Properties of biautomatic groups:

1. Solvable word problem
2. Solvable conjugacy problem in exponential time
3. Solvable subgroups are virtually abelian.

Mosher: $\mathrm{Mod}(S)$ is automatic.

Thm: $\mathrm{Mod}(S)$ is biautomatic.

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Strategy of proof:

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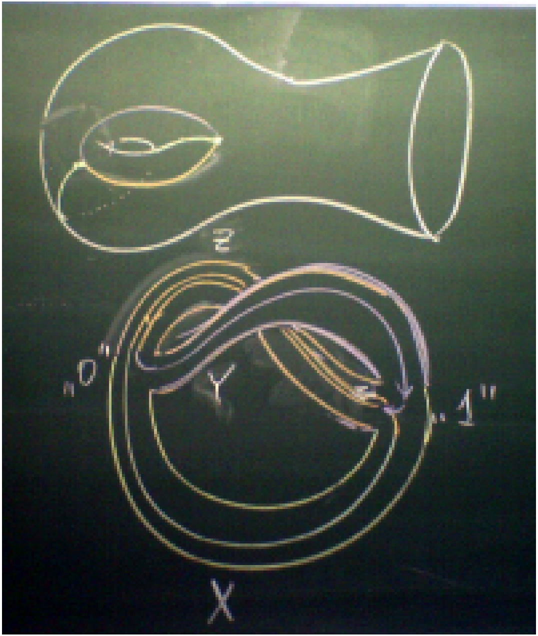
6. Show that the paths correspond to words of a regular language in a finite alphabet.

Part one: The directed graph

Def: A *complete generic train track* on S is an embedded graph in S s.th.

1. all vertices (=switches) are trivalent
2. there is a tangent line everywhere
3. complementary components are trigons
4. there is a journey from every direction to every direction
5. transverse recurrence


„0"
„1"
X
Y

Def: *Train track complex* $\mathcal{TT}$: vertices = (isotopy classes of) complete train tracks on S
directed edges:

Two facts:

1. $\mathcal{TT}$ is connected.
2. $\mathrm{Mod}(S)$ acts properly and cocompactly on $\mathcal{TT}(S)$.

Directed paths: *splitting sequences*

Part 2: Quasigeodesics

Directions for directed paths

Fix a hyperbolic metric on S.

Def: A *complete geodesic lamination* λ is a closed subset of S foliated into simple geodesics s.th.

a) complementary regions are ideal triangles

b) λ is a Hausdorff limit of a sequence of simple closed geodesics.

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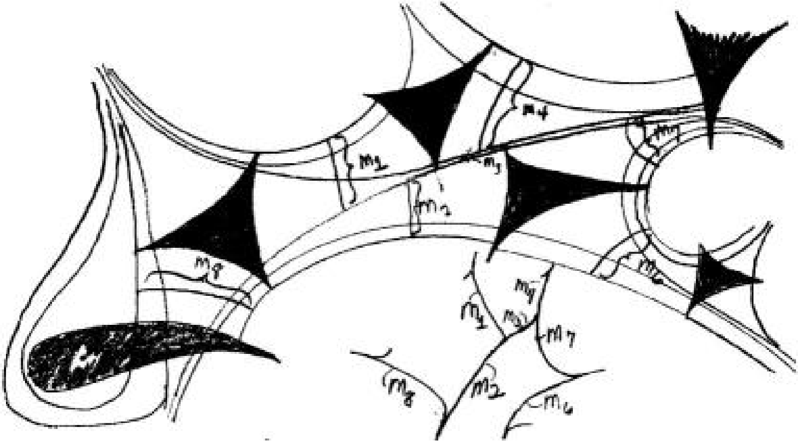
b) λ is a Hausdorff limit of a sequence of simple closed geodesics.

Simple Fact: The space $\mathcal{CL}$ of all complete geodesic laminations with the Hausdorff topology is a compact metrizable totally disconnected $\mathrm{Mod}(S)$-space.



Def: A train track or a geodesic lamination λ is *carried* by a train track τ ($\lambda \prec \tau$) if there is $F : S \to S$ of class C^1 homotopic to Id, $dF|T\lambda$ of maximal rank, with $F(\lambda) \subset \tau$.

Important facts:

1. Every complete train track τ carries a complete geodesic lamination λ.
2. If λ is carried by τ, then for a large branch e there is a unique choice of a split at e such that the split track carries λ.

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$E(\tau, \lambda) =$ the set of all endpoints of directed simplicial arcs issuing from τ which carry λ.

Proposition:

1. $E(\tau, \lambda)$ with its intrinsic path metric d_E is isometric to the one-skeleton of a $\mathrm{CAT}(0)$-cubical euclidean complex of uniform polynomial growth.
2. $E(\tau, \lambda)$ is not contained in a quasi-flat of $\mathcal{TT}$ of strictly bigger dimension.
3. Directed paths are geodesics in $E(\tau, \lambda)$.
4. The inclusion $(E(\tau, \lambda), d_E) \to \mathcal{TT}$ is an L-quasi-isometric embedding for some universal L.

Means: Directed edge-paths are uniform quasi-geodesics in $\mathcal{TT}$.

Digression: Furstenberg boundary

G semi-simple Lie group of non-compact type, $P < G$ maximal parabolic subgroup $\Rightarrow$
the *Furstenberg boundary* G/P is a compact metrizable G-space.
The G-action is *amenable*

G/P parametrizes *Weyl chambers*= maximal flat cones in G/K.

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Theorem:

1. The $\mathrm{Mod}(S)$-action on the space $\mathcal{CL}$ of complete geodesic laminations on S is amenable.
2. $\mathcal{CL}$ parametrizes maximal cones of uniform polynomial growth.

Application:

Thm: Let $n \geq 2$, G_i locally compact topological groups ($i \leq n$), $\Gamma < G = G_1 \times \cdots \times G_n$ an irreducible lattice, $\rho : \Gamma \to \mathrm{Mod}(S)$ a homeomorphism.
Then $\rho(\Gamma) < H_0 \times H_1 < \mathrm{Mod}(S)$, H_0 virtually abelian, H_1 contains a finite normal subgroup K, ρ induces a continuous homomorphism $\tilde{\rho} : G \to H_1/K$.

True also for $\mathrm{Mod}(S)$-valued cocycles.

Earlier results: Kaimanovich-Masur, Farb-Masur, Bestvina-Fujiwara,.....

Part 4: Synchronization:

Difficulty in discrete product spaces ($\mathbb{Z} \times \mathbb{Z}$):
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There are many geodesics connecting two points.

Only way to obtain quasiconvex bicombing:
Pass along the diagonal as long as possible
continue passing along the diagonal of hyperplanes

Fundamental difficulty:
A subtrack of a train track is a subset which is itself a train track.

Example: A simple closed embedded curve

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Example: A simple closed embedded curve

There are infinite sequences working along a subtrack (e.g. filling a proper subsurface) and dragging along the rest of the train track
$\Rightarrow$ synchronization with *multi-splits* is needed.

Example: Dehn twists

Part 6: The biautomatic structure:

Easy fact: A *numbered train track* is a train track with a fixed numbering of the branches.
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Automatic processing of the train track complex (almost gives a biautomatic structure for $\mathrm{Mod}(S)$):

Alphabet $A=$ combinatorial types of complete numbered train tracks on S.

States: A beginning state, a rejecting state
Combinatorial types of numbered train tracks with traffic control system
a finite set of additional states $s_1, \dots, s_k$.

Functioning of the machine:

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Summary:

The above constructions provide a rather explicit geometric understanding of $\mathrm{Mod}(S)$ which can be automatically processed.

Outlook and open question:

1. Do mapping class groups have property τ?
2. Giving any finite group G, is it possible to detect whether the mapping class group virtually surject onto G?
3. Are mapping class groups linear?
4. Do mapping class groups virtually surject onto $\mathbb{Z}$?
5. Do mapping class groups have the Haagerup property?
6. Do mapping class groups have f.g. purely pseudo-Anosov subgroups which are not virtually free?
7. Moving train tracks are moving "triangulations" of the surface. Is this useful for TQFT?
8. Is it possible to reconcile analysis and combinatorics by translating the biautomatic structure to the moduli space of quadratic differentials?
9. Is it possible to reconcile analysis and combinatorics by giving a combinatorial proof of "not T"?