1. We define $f''(x)$ to be the derivative of the derivative of $f(x)$.
(a) Let $f(x) = x^2$. Find $f''(x)$.
(b) Let $g(x) = \sin x$. Find $g''(x)$.

2. (Alternative form of the Mean Value Theorem) Let f be a function that is differentiable on $[a, b]$ and let M_{max} and M_{min} be the absolute maximum and minimum values attained by f' on the interval $[a, b]$. Show that for any x in the interval $[a, b]$, we have

$$f(x) - f(a) < M_{max}(x - a)$$

and

$$f(x) - f(a) > M_{min}(x - a)$$

3. (Hard but interesting - this is called Taylor's theorem with remainder) Let f and f' be differentiable on $[a, b]$. Let T_{max} be the absolute maximum value attained by f'' on the interval $[a, b]$. Let $L(x)$ be the best linear approximation to $f(x)$ at $x = a$. Show that

$$f(x) - L(x) < T_{max}(x - a)^2.$$

4. With the same assumptions, switching things around, we can show that if T_{min} is the absolute minimum value attained by f'' on the interval $[a, b]$, then

$$f(x) - L(x) > T_{min}(x - a)^2.$$

Then, letting M equals the larger of $|T_{max}|$ and $|T_{min}$, we can show that

$$|f(x) - L(x)| < T(x - a)^2.$$

Use this to show that if $L(x)$ is the best linear approximation to $f(x)$ at $x = 1$, then on the interval $[1, 1.2]$, we always have

$$|f(x) - L(x)| < .04.$$