MATH 4200/6200 FINAL EXAM, FALL 2008

There are 140 points on the exam, distributed as indicated in parentheses. In a multi-part problem, if you don't get one of the parts, you may still assume the result of that part in each successive part. You may consult only Munkres, your class notes, your homeworks, and the homework solutions; the solutions will be posted for the duration of the exam at http://math.uga.edu/~usher/4200/solutions.pdf

The exam is due in my office (321C) by 3pm Friday, December 12; you can slide it under my door if I'm not there.

1. Let X be the subspace $(\mathbb{R}\times\{0\})\cup(\mathbb{R}\times\{1\})$ of $\mathbb{R}^2$ (with its standard topology). Define an equivalence relation on X by saying that two points (x, i) and (y, j) are equivalent if and only if either they are equal or $x = y \neq 0$ (it should be fairly obvious that this is an equivalence relation, but you need not prove it). Let X^* be the set of equivalence classes for this equivalence relation, and equip X^* with the quotient topology.

(a) (0 points, but probably a good idea before proceeding) Identify X^* with a more simply-described set, and determine what the open subsets in X^* are in terms of this identification.

(b) (10 points) Which of the separation axioms T_1, T_2, T_3, T_4 are satisfied by X^*?

(c) (10 points) Prove or disprove that X^* is connected.

2 (10 points). Let X be a set and, for all α belonging to some nonempty index set J, let $f_\alpha\colon X \to A_\alpha$ be a function, where A_α is a topological space. Let

$$\mathcal{S} = \{f_\alpha^{-1}(U_\alpha) | \alpha \in J,\, U_\alpha \subset A_\alpha \text{ is open}\}.$$

Prove that $\mathcal{S}$ is a subbasis for a topology $\mathcal{T}$ on X such that

(i) every $f_\alpha\colon X \to A_\alpha$ is continuous (as a map from the topological space $(X, \mathcal{T})$ to A_α), and

(ii) If $\mathcal{T}'$ is a topology on X also satisfying (a) (with $\mathcal{T}$ replaced by $\mathcal{T}'$), then $\mathcal{T} \subset \mathcal{T}'$.

3. A topological space X is said to be T_0 if for all distinct points $x, y \in X$, there is either (but not necessarily both) an open set containing x but not y or an open set containing y but not x.

(a) (5 points) Prove that a space that is both T_3 and T_0 is also T_2. (Recall that we use T_3 to just mean that any point and any closed set not containing that point can be separated by open sets, so that in principle (in our convention) a T_3 space need not be T_1).

(b) (5 points) Give an example of a topological space X which is T_0 but in which, for each $x \in X$, the set $\{x\}$ is not closed.

(c) (5 points) Suppose that Y has the indiscrete topology, that X is a T_0 space, and that $f\colon Y \to X$ is continuous. Prove that f is constant.

4 (15 points). Prove the following theorem (which is known as the *contractive mapping principle*):

Theorem 0.1. *Let (X, d) be a complete metric space, $0 < r < 1$, and let $f : X \to X$ be a function with the property that, for all $x, y \in X$, one has*

$$d(f(x), f(y)) \leq rd(x, y).$$

Then there exists a unique point $x^* \in X$ such that $f(x^*) = x^*$. In fact, if x_1 is any point of X and if for $n \geq 1$ we set $x_{n+1} = f(x_n)$, then $\lim_{n\to\infty} x_n$ exists and is equal to x^*.

5. (a) (7 points) If D is a dense subspace of a topological space X, and if $U \subset X$ is open, prove that $\overline{D \cap U} = \bar{U}$.

(b) (3 points) Give an example showing that part (a) need not hold if we drop the assumption that U is open.

(c) (10 points) Let X be a Hausdorff space and let $Y \subset X$ be a dense subset of X which is locally compact with respect to the subspace topology. Prove that Y is an open subset of X. *Hint*: First show that any point of y has an open-in-Y neighborhood whose closure in X is contained in Y, and then apply (a).

6. (a) (10 points) Prove that $C([0, 1], [0, 1])$ (*i.e.*, the space of continuous functions from $[0, 1]$ to $[0, 1]$) is not locally compact when given the uniform topology.

(b) (10 points) Prove or disprove that $C([0, 1], [0, 1])$ is compact when it is given the subspace topology induced by the product topology on $[0, 1]^{[0,1]}$.

7 (10 points). Let X and Y be topological spaces such that X is first countable, and let $f \colon X \to Y$ be a function with the property that, whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in X and $x \in X$ with $x_n \to x$, we have $f(x_n) \to f(x)$ in Y. Prove that f is continuous.

8 (15 points). Let X be a complete metric space and let $\mathcal{F}$ be a collection of continuous functions $f \colon X \to \mathbb{R}$. Prove that one of the two following alternatives must hold:

(i) There is $x \in X$ such that the set $\{f(x)|f \in \mathcal{F}\}$ is unbounded (*i.e.*, for any M there is $f \in \mathcal{F}$ such that $|f(x)| \geq M$); or

(ii) For some nonempty open set $U \subset X$, there is $M > 0$ such that $|f(x)| \leq M$ for *all* $x \in U$ and $f \in \mathcal{F}$.

Hint: Apply Baire's theorem, making use of the sets

$$\{x \in X | |f(x)| \leq m \text{ for all } f \in \mathcal{F}\}.$$

9 (15 points). If $A \subset X$, a function $f \colon X \to [0, 1]$ is said to *vanish precisely along* A if $f(x) = 0$ for all $x \in A$ while $f(x) > 0$ for $x \notin A$. If X is a normal topological space, prove that there exists a continuous function $f \colon X \to [0, 1]$ which vanishes precisely along A if and only if A is a closed G_δ set (recall that a G_δ set is a set which can be written as a countable intersection of open sets).