Math 8230, Fall 2009: Problem Set 2. Due Tuesday, September 15

1. Coisotropic subspaces

(a) Let (V, ω) be a symplectic vector space, and let $W \leq V$ be a subspace of codimension one. Prove that W is coisotropic.

(b) Let (V, ω) be a symplectic vector space and let $W \leq V$ be a coisotropic subspace. If $x \in W$, use the notation $[x]$ to denote the equivalence class of x in the quotient vector space W/W^{ω}. Prove that the formula

$$\bar{\omega}([x],[y])$$

gives a well-defined function on $(W/W^{\omega}) \times (W/W^{\omega})$, and that $(W/W^{\omega}, \bar{\omega})$ is a symplectic vector space.

(c) A *presymplectic vector space* is a pair (W, ω_1) where W is a vector space equipped with a skew-symmetric bilinear form ω_1. If (W, ω_1) is a presymplectic vector space, let

$$N = \{v \in W | (\forall w \in W)(\omega_1(v, w) = 0)\},$$

and choose an arbitrary subspace $U \leq W$ satisfying $W = U \oplus N$. Define

$$V = W \oplus N^* = U \oplus N \oplus N^* \text{ and } \omega\colon V \times V \to \mathbb{R}$$

by

$$\omega((u_1, n_1, \alpha_1), (u_2, n_2, \alpha_2)) = \omega_1(u_1, u_2) + \alpha_2(n_1) - \alpha_1(n_2).$$

Prove that:

(i) (V, ω) is a symplectic vector space,
(ii) $W \leq V$ is a coisotropic subspace with $W^{\omega} = N$
(iii) The map $(U, \omega|_U) \to (W/W^{\omega}, \bar{\omega})$ defined by $u \mapsto [u]$ is a linear symplectomorphism, and
(iv) If (V', ω') is some other symplectic vector space such that $W \leq V'$ is coisotropic and $\omega'|_W = \omega|_W$, then there is a linear symplectomorphism $A\colon (V, \omega) \to (V', \omega')$ such that $A|_W$ is the identity. (It might help to prove the following lemma: if (X, Ω) is a symplectic vector space and $\{e_1, \ldots, e_k\}$ is a basis for a Lagrangian subspace L, then there is a standard symplectic basis of the form $\{e_1, \ldots, e_k, f_1, \ldots, f_k\}$ for X with $\Omega(e_i, f_j) = \delta_{ij}$ and $\Omega(f_i, f_j) = 0$).

Remark 1.1. *Note that, in the case where W is Lagrangian, we will have $U = 0$ and $\omega|_W = 0$ above, and so the last part shows that V' is linearly symplectomorphic, by a map restricting to W as the identity, to $W \oplus W^*$ with the standard symplectic form $\omega((w_1, \alpha_1), (w_2, \alpha_2)) = \alpha_2(w_1) - \alpha_1(w_2)$.*

2. Line complexes and the origin of the word "symplectic"

According to a definition from 19th century projective geometry [1], a *line complex* is a hypersurface in the space of lines in real projective space $\mathbb{R}P^{k-1}$ for some k; in turn, this space of lines can be identified with the Grassmannian $G_{2,k}(\mathbb{R})$ of two-dimensional planes in $\mathbb{R}^k$. If $k = 2n$, then the collection $\mathcal{C}$ of 2-planes P in $\mathbb{R}^{2n}$ such that $\omega_0|_P = 0$ is an example of a line complex.

Let

$$G = \{A \in SL(2n; \mathbb{R}) | (\forall P \in \mathcal{C})(A(P) \in \mathcal{C})\}.$$

Prove that if n is odd then $G = Sp(2n)$, and that if n is even then G is disconnected but that its identity component is $Sp(2n)$.

Remark 2.1. *Thus, one arrives at $Sp(2n)$ by considering elements that* preserve the line complex *$\mathcal{C}$. This initially led Hermann Weyl to christen it the "complex group;" however, as he wrote in [3], "The name 'complex group' formerly advocated by me in allusion to line complexes...has become more and more embarrassing to me through collision with the word 'complex' in the connotation of complex number. I therefore propose to replace it by the corresponding Greek adjective 'symplectic'." (The Greek prefix 'sym' and the Latin prefix 'com' both mean 'together' and the Latin 'plexus' and the Greek* *'plektos' both derive from verbs meaning 'folded' or 'plaited.') My main source for this problem was an interesting survey article by Alan Weinstein from 1981* [2]*, in which it is stated that, "According to the Oxford English Dictionary, the only use of 'symplectic' in English is with reference to a certain bone in the head of fish." However, nowadays the OED includes the mathematical usage as well.*

3. The quaternions and the symplectic group

There is another group that, unfortunately, is sometimes given the same name ($Sp(2n)$) that we've been using for the group of linear symplectomorphisms of $\mathbb{R}^{2n}$. This problem will describe that group (we'll call it $\overline{Sp}(n)$) and its relation to $Sp(2n)$.

Let $\mathbb{H}$ denote the quaternions, *i.e.*, the division ring consisting of symbols $a + ib + jc + kd$ where $a, b, c, d \in \mathbb{R}$ and the operations are obtained by extending $\mathbb{R}$-linearly from $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$, $i^2 = j^2 = k^2 = -1$. *Conjugation* on $\mathbb{H}$ or on $\mathbb{H}^n$ (denoted as usual with a bar) is the $\mathbb{R}$-linear operator induced by switching the signs of each of i, j, k. If $v, w \in \mathbb{H}^n$, the inner product of v and w is, by definition $\langle v, w\rangle = \bar{w}^T v$.

(a) If M is a $n \times n$ matrix with coefficients in $\mathbb{H}$, we can write $M = A + jB$ where A and B are $n \times n$ complex matrices. Identify $M = A + jB$ with the $2n \times 2n$ complex matrix

$$\begin{pmatrix} A & -\bar{B} \\ B & \bar{A} \end{pmatrix}.$$

Prove that M preserves the inner product on $\mathbb{H}^n$ (i.e. $\langle Mv, Mw\rangle = \langle v, w\rangle$) if and only if

$$\begin{pmatrix} A & -\bar{B} \\ B & \bar{A} \end{pmatrix} \text{ is a unitary } 2n \times 2n \text{ matrix.}$$

Accordingly, we let

$$\overline{Sp}(n) = \left\{ U \in U(2n) | U = \begin{pmatrix} A & -\bar{B} \\ B & \bar{A} \end{pmatrix} \right\};$$

thus (under the above identification of $n \times n$ $\mathbb{H}$-valued matrices with complex matrices), $\overline{Sp}(n)$ can be thought of as the quaternionic version of the unitary group.

(b) Let $Sp(2n; \mathbb{C})$ denote the *complex* symplectic group, *i.e.* those $2n \times 2n$ matrices S with complex entries that satisfy $S^T J_0 S = J_0$. Prove that

$$\overline{Sp}(n) = Sp(2n; \mathbb{C}) \cap U(2n).$$

(c) (Extra credit; only do this if you know some Lie theory) Prove that the Lie algebras of $Sp(2n)$ and $\overline{Sp}(n)$ are both real forms of the Lie algebra of $Sp(2n; \mathbb{C})$ (recall that if $\mathfrak{g}$ is a Lie algebra over $\mathbb{C}$ and $\mathfrak{h}$ is a Lie algebra over $\mathbb{R}$, $\mathfrak{h}$ is called a real form of $\mathfrak{g}$ if $\mathfrak{g}$ is isomorphic to $\mathfrak{h} \otimes_{\mathbb{R}} \mathbb{C}$. In particular this holds if $\mathfrak{h} \subset \mathfrak{g}$ and $\mathfrak{g} = \mathfrak{h} \oplus i\mathfrak{h}$).

(d) Prove that $Sp(2n)$ is not homeomorphic to $\overline{Sp}(n)$.

References

[1] J. Plücker. *Neue geometrie des raumes gegründet auf die Betrachtung der geraden linie als raumelement.* Teubner, Leipzig, 1868.
[2] A. Weinstein. *Symplectic geometry.* Bulletin of the AMS. **5** (1981), no. 1, 1–13.
[3] H. Weyl. *The classical groups: their invariants and representations.* Princeton University Press, Princeton, 1946.