Math 8230, Fall 2009: Problem Set 2. Solutions to selected problems

1. (c) (iv) Let (V', ω') be a symplectic vector space with coisotropic subspace W having $\omega'|_W = \omega|_W$. (Here, where $W = U \oplus N$, ω is the symplectic form on $V = U \oplus N \oplus N^*$ given by $\omega((u_1, n_1, \alpha_1), (u_2, n_2, \alpha_2)) = \omega_1(u_1, u_2) + \alpha_2(n_1) - \alpha_1(n_2)$.) In particular, U is a symplectic subspace of both (V, ω) and (V', ω'). Consider the symplectic orthogonal complement $U^{\omega'} \subset V'$. Since U is symplectic, so is $U^{\omega'}$, and $U \cup U^{\omega'} = \{0\}$.

An earlier part showed that $W^\omega = N$. Meanwhile, $W^{\omega'} \subset W$ since W is coisotropic in (V, ω'), and so since $\omega'|_W = \omega|_W$ we obtain that $W^{\omega'} = N$. Hence

$$\dim V' = \dim W + \dim N = \dim V.$$

Thus

$$\dim U^{\omega'} = \dim V' - \dim U = \dim V - \dim U = \dim U^\omega = 2 \dim N.$$

Now since $N \subset W$ $\omega'|_N = \omega|_N = 0$, and if $u \in U$, $n \in N$ we have $\omega'(u, n) = 0$. Thus $N \subset U^{\omega'}$, N has half the dimension of V', and ω' vanishes on N. So N is a Lagrangian subspace of $U^{\omega'}$.

Now assume the lemma that is stated in the hint, which we'll prove momentarily. Choose a basis $\{e_1, \ldots, e_k\}$ of N, which by the lemma extends to a basis $\{e_1, \ldots, e_k, f_1, \ldots, f_k\}$ of $U^{\omega'}$, with $\omega'(e_i, f_j) = \delta_{ij}$ and $\omega'(f_i, f_j) = 0$. Where $\{e_1^*, \ldots e_k^*\}$ is the dual basis of N^* (thus $e_i^*(e_j) = \delta_{ij}$), we have

$$\omega'(e_i, f_j) = \delta_{ij} = \omega(e_i, e_j^*) \text{ and } \omega'(f_i, f_j) = \omega(e_i^*, e_j^*) = 0.$$

As a result, the linear map $A\colon V \to V'$ defined by setting $A|_W$ equal to the identity and $Ae_i^* = f_i$ is a linear symplectomorphism (since the f_i belong to $U^{\omega'}$ by construction).

It remains only to prove the lemma. So let (X, Ω) be a symplectic vector space containing the Lagrangian subspace L, with $\{e_1, \ldots, e_k\}$ a basis for L. Assume by induction (starting at the vacuous case $l = 0$) that we've constructed $f_1, \ldots, f_l$ so that $\Omega(e_i, f_j) = \delta_{ij}$ and $\Omega(f_i, f_j) = 0$ whenever these are defined (N.B.: the condition $\Omega(e_i, f_j) = \delta_{ij}$ includes cases where $i > l$). Let

$$Y_l = span\{e_i : i \neq l + 1\}.$$

So Y_l is a codimension-one subspace of L; hence L^ω is a codimension-one subspace of Y_l^ω. Let f_{l+1}^0 be an arbitrary element of $Y_l^\omega \setminus L^\omega$, so $\omega(e_i, f_{l+1}^0) \neq 0$ iff $i = l + 1$. Now for $1 \leq i \leq l$, inductively put

$$f_{l+1}^i = f_{l+1}^{i-1} - \omega(f_{l+1}^{i-1}, f_i) e_i.$$

So if $j \neq i$ then $\omega(f_{l+1}^i, f_j) = \omega(f_{l+1}^{i-1}, f_j)$, while since $\omega(e_i, f_j) = 1$ by induction we see $\omega(f_{l+1}^i, f_i) = 0$. Also since ω vanishes on L and since $\omega(f_{l+1}^0, e_j) = 0$ for $j \neq l + 1$ we find $\omega(f_{l+1}^i, e_j) = 0$ for $j \neq l + 1$.

At the end of the induction on i we obtain f_{l+1}^l so that $\omega(f_{l+1}^l, f_j) = 0$ for $j \leq l$, $\omega(f_{l+1}^l, e_j) = 0$ for $j \neq l + 1$, and $\omega(f_{l+1}^l, e_{l+1}) \neq 0$. So putting $f_{l+1} = -f_{l+1}^l / \omega(f_{l+1}^l, e_{l+1})$ completes the inductive step.

2. Write the standard basis for $\mathbb{R}^{2n}$ as $\{e_1, \ldots, e_{2n}\}$, so ω_0 is given by $\omega_0(e_i, e_j) = \pm 1$ when $j = i \pm n$ and $\omega_0(e_i, e_j) = 0$ otherwise.

Let $A \in G$. If $|i - j| \neq n$, then ω_0 vanishes on the 2-plane spanned by e_i and e_j; since $A \in G$ ω_0 also vanishes on the image of this 2-plane under A. We thus have, for $|i - j| \neq n$,

$$0 = \omega(Ae_i, Ae_j) = (Ae_j)^T J_0 A e_i = e_j^T (A^T J_0 A) e_i = (A^T J_0 A)_{ji}.$$

Now suppose $1 \leq i \leq n - 1$. Then

$$\omega_0(e_i - e_{i+1}, e_{i+n} + e_{i+1+n}) = \omega_0(e_i, e_{i+n}) - \omega_0(e_{i+1}, e_{i+1+n}) = 0.$$

Hence

$$\begin{aligned} 0 &= \omega_0(A(e_i - e_{i+1}), A(e_{i+n} + e_{i+1+n})) = (e_{i+n} + e_{i+1+n})^T (A^T J_0 A)(e_i - e_{i+1}) \\ &= (A^T J_0 A)_{i+n,i} + (A^T J_0 A)_{i+1+n,i} - (A^T J_0 A)_{i+n,i+1} - (A^T J_0 A)_{i+1+n,i+1} = (A^T J_0 A)_{i+n,i} - (A^T J_0 A)_{i+1+n,i+1}, \end{aligned}$$

where the last inequality follows from the fact that we've already shown $(A^TJ_0A)_{ij}=0$ when $|i-j|\neq n$. Hence, for some $\alpha\in\mathbb{R}$, we have $(A^TJ_0A)_{i+n,i}=\alpha$. Meanwhile, note that $(A^TJ_0A)^T=A^TJ_0^TA=-A^TJ_0A$; thus, for this same $\alpha\in\mathbb{R}$ we have $(A^TJ_0A)_{i,i+n}=-\alpha$.

This determines all entries of A^TJ_0A, proving that $A^TJ_0A=\alpha J_0$ for some $\alpha\in\mathbb{R}$ whenever $A\in G$. Thus

$$G\subset\{A\in SL(2n;\mathbb{R})|(\exists\alpha\in\mathbb{R})A^TJ_0A=\alpha J_0\}$$

Conversely, if $A\in SL(2n,\mathbb{R})$ and $A^TJ_0A=\alpha J_0$, suppose that $P\subset\mathbb{R}^{2n}$ is a 2-plane on which ω_0 vanishes. Thus where $\{v,w\}$ is a basis for P we have $\omega_0(v,w)=w^TJ_0v=0$. But then $\omega_0(Av,Aw)=v^TA^TJ_0Aw=\alpha v^TJ_0w=0$. Thus ω_0 also vanishes on $A(P)$ (which is a 2-plane since A is invertible). P was an arbitrary element of the line complex $\mathcal{C}$, so this proves that $A\in G$. So

$$G=\{A\in SL(2n;\mathbb{R})|(\exists\alpha\in\mathbb{R})A^TJ_0A=\alpha J_0\}.$$

Evidently we have $Sp(2n)\subset G$, since $Sp(2n)$ consists of those matrices A with $A^TJ_0A=J_0$. On the other hand, if $A\in G$ with $A^TJ_0A=\alpha J_0$, we have $A^*\omega_0=\alpha\omega_0$, and so $A^*\omega_0^n=\alpha^n\omega_0^n$. But by the definition of G we have $\det A=1$, so that $A^*\omega_0^n=\omega_0^n$. Thus $\alpha^n=1$. If n is odd, then this forces $\alpha=1$, proving that $G=Sp(2n)$ when n is odd. If n is even, we conclude that $\alpha=\pm1$. In this case, define a map $F\colon G\to\{-1,1\}$ by setting $F(A)=\alpha$ where $A^TJ_0A=\alpha J_0$. Obviously $Sp(2n)=F^{-1}(\{1\})$, and in particular $F^{-1}(\{1\})$ is connected since we saw in class that $Sp(2n)$ is connected. Meanwhile the matrix A defined by

$$A_{ij}=\begin{cases}\delta_{ij} & 1\leq i\leq n\\ -\delta_{ij} & n+1\leq i\leq 2n\end{cases}$$

has determinant 1 (for n even) and obeys $A^TJ_0A=-J_0$. Thus $F\colon G\to\{-1,1\}$ is surjective. Since F is continuous, it follows that G is disconnected, with its identity component equal to $F^{-1}(\{1\})=Sp(2n)$ (since we've already noted that the latter is connected).

3. (c) Linearizing the equation $A^TJ_0A=J_0$ around $A=I$ (*i.e.*, taking the derivative of both sides as a function of t where A is replaced by $I+tX$ and setting $t=0$) gives rise to the equation $X^TJ_0+J_0X=0$. Thus $Sp(2n)$ has Lie algebra

$$Lie(Sp(2n))=\{X\in M_n(\mathbb{R})|X^TJ_0+J_0X=0\}.$$

The same argument with complex coefficients shows that

$$Lie(Sp(2n;\mathbb{C}))=\{Z\in M_n(\mathbb{C})|Z^TJ_0+J_0Z=0\}.$$

So obviously $Lie(Sp(2n))\oplus i\cdot Lie(Sp(2n))=Lie(Sp(2n;\mathbb{C}))$; thus $Lie(Sp(2n))$ is a real form of $Lie(Sp(2n;\mathbb{C}))$.

Meanwhile, since $\overline{Sp}(n)=Sp(2n;\mathbb{C})\cap U(2n)$, we have

$$Lie(\overline{Sp}(n))=\{X\in Lie(Sp(2n;\mathbb{C}))|X^*+X=0\}$$

where $*$ denotes Hermitian adjoint. Now $Lie(Sp(2n;\mathbb{C}))$ is closed under complex scalar multiplication, and of course one has $X^*+X=0$ if and only if $(iX)^*-(iX)=0$. Thus

$$i\cdot Lie(\overline{Sp}(n))=\{X\in Lie(Sp(2n;\mathbb{C}))|X^*-X=0\}.$$

But $M_n(\mathbb{C})$ is the direct sum of the space of Hermitian matrices with the space of skew-Hermitian matrices (as these spaces obviously have trivial intersection, and if M is any complex square matrix then we can write $M=\frac{1}{2}(M+M^*)+\frac{1}{2}(M-M^*)$ where the first term is Hermitian and the second is skew-Hermitian). Hence

$$Lie(\overline{Sp}(n))\oplus i\cdot Lie(\overline{Sp}(n))=Lie(Sp(2n;\mathbb{C})),$$

and so $Lie(\overline{Sp}(n))$ is a real form of $Lie(Sp(2n;\mathbb{C}))$.

(d) $Sp(2n)$ is not homeomorphic to $\overline{Sp}(n)$ because the former is noncompact while the latter is compact (of course, the topologies on these spaces are to be understood as coming from their embeddings as subspaces of $\mathbb{R}^{4n^2} = \mathbb{H}^{n^2}$ given by their entries). To see this, first consider $Sp(2n)$. For any $m \in \mathbb{R} \setminus \{0\}$, the matrix A_m which has all entries equal to those the identity except $(A_m)_{11} = m$ and $(A_m)_{n+1,n+1} = 1/m$ belongs to $Sp(2n)$, and so the continuous map $Sp(2n) \to \mathbb{R}$ which sends A to A_{11} has unbounded image. But if $Sp(2n)$ were compact, this map would need to have compact image, a contradiction.

Now consider $\overline{Sp}(n)$. If $A \in \overline{Sp}(n)$, then for all $m \in \{1, \ldots, n\}$ we have, where e_m is the mth standard basis element of $\mathbb{H}^n$,

$$1 = \langle e_m, e_m \rangle = \langle Ae_m, Ae_m \rangle = \bar{e}_m^T \bar{A}^T A e_m = (\bar{A}^T A)_{mm}) = \sum_{r=1}^{n} \bar{A}_{rm} A_{rm} = \sum_{r=1}^{n} |A_{rm}|^2.$$

In particular, if $A \in \overline{Sp}(n)$ this shows that, for all r, m, the entry A_{rm} belongs to the closed unit disk $\overline{D}^4 \subset \mathbb{H}$. Hence $\overline{Sp}(n) \subset (\overline{D}^4)^{n^2}$, which is a compact subset of $\mathbb{H}^{n^2}$. Furthermore, since the map $A \mapsto \bar{A}^T A$ is obviously continuous as a map from $\mathbb{H}^{n^2}$ to itself and $\overline{Sp}(n)$ is the preimage of the identity under this map, it follows that $\overline{Sp}(n)$ is a closed subset of $\mathbb{H}^{n^2}$. Thus $\overline{Sp}(n)$ is closed and bounded, and therefore compact.