Math 8230, Fall 2009: Problem Set 6. Due Thursday, November 19

1. Symplectic fiber bundles

Let (F, σ) be a symplectic manifold. A *symplectic fibration* modeled on (F, σ) consists of a surjective submersion $\pi\colon E \to B$ of smooth manifolds, together with an open cover $\{U_\alpha\}_{\alpha \in A}$ of B and local trivializations $\phi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times F$ (which, as usual, are diffeomorphisms which map each fiber $\pi^{-1}(\{x\})$ to $\{x\} \times F$), such that, when $x \in U_\alpha \cap U_\beta$, the transition map $\phi_\beta \circ \phi_\alpha^{-1}$ restricts to $\{x\} \times F$ as a *symplectomorphism* of the symplectic manifold (F, σ).

(a) Suppose that $\pi\colon E \to B$ is a smooth fiber bundle and that $\tau \in \Omega^2(E)$ is a two-form which restricts as a symplectic form to each fiber and which is vertically closed (*i.e.* if $v, w \in \ker \pi_*$ then $d\tau(v, w, \cdot) = 0$). Prove that E admits the structure of a symplectic fibration.

(b) (cf. McDuff-Salamon Exercise 6.19) Conversely, given a symplectic fibration $\pi\colon E \to B$ modeled on (F, σ), construct a vertically closed $\tau \in \Omega^2(E)$ such that τ restricts to each fiber of π as a symplectic form.

(c) As discussed in class, a form τ as in (b) induces a *connection*, *i.e.*, a subbundle T^hE of TE which is complementary to $\ker \pi_*$, defined by

$$T^hE = \{v \in TE | (\forall w \in \ker \pi_*)(\tau(v, w) = 0)\}.$$

Call τ *trivial* if there is a diffeomorphism $\Phi\colon E \to F \times B$, restricting to each $\pi^{-1}(\{b\})$ as a symplectomorphism $\pi^{-1}(\{b\}) \to F \times \{b\}$, such that $\Phi^*(\pi_1^*\sigma) = \tau$ (where $\pi_1\colon F \times B \to F$ is the projection).[1] Prove that if τ is trivial then the commutator of any two vector fields tangent to T^hE must also be tangent to T^hE. Use this to construct a closed but nontrivial τ on the trivial symplectic fibration $F \times S^1 \times S^1 \to S^1 \times S^1$.

2. The symplectic cut

Let (M, ω) be a symplectic manifold and $H\colon M \to \mathbb{R}$ a smooth function whose Hamiltonian vector field X_H induces an action of $S^1 = \mathbb{R}/\mathbb{Z}$. If $\epsilon \in \mathbb{R}$ is any regular value of H such that S^1 acts freely on $H^{-1}(\epsilon)$, recall the construction of the symplectic cut $M_{\geq\epsilon}$: define $\nu\colon \mathbb{C} \times M \to \mathbb{R}$ by $\nu(z, m) = H(m) - \pi|z|^2$, so that the Hamiltonian vector field of ν also induces an S^1 action, and let $M_{\geq\epsilon}$ be the symplectic reduction $\nu^{-1}(\epsilon)/S^1$, with its induced symplectic form ω_ϵ. As noted in class, M_ϵ contains as an open dense subset the set $M_{>\epsilon} = H^{-1}((\epsilon, \infty)) \subset M$. **Prove** that the restriction to $M_{>\epsilon}$ of ω_ϵ coincides with the restriction to $M_{>\epsilon}$ of ω.

[1] Note that obviously τ has to be closed, not just vertically closed, in order to be trivial.