MATH 2250 Exam

February 9, 2009

Name ____

Answer every question on the exam—there is no penalty for guessing. Calculators and similar aids are not allowed. There are a total of **60 points** possible.

1 (8 parts, 4 points each). Find the derivatives of the following functions:

a) $f(x) = 2x + \frac{3}{x^2}$.

$$f'(x) = 2 + 3(-2)x^{-3} = 2 - \frac{6}{x^3}$$

b) $f(x) = x^2 e^x$.

By the product rule,

$$f'(x) = x^2 \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(x^2) = x^2 e^x + 2xe^x.$$

c) $f(x) = \frac{\sin x}{x^2+1}$.

By the quotient rule,

$$f'(x) = \frac{(x^2+1)\cos x - 2x \sin x}{(x^2+1)^2}.$$

d) $f(x) = \tan(x) + \cos(x)$

$$f'(x) = \sec^2 x - \sin x.$$

e) $f(x) = e^{-\sqrt{x}}$.

$$f'(x) = e^{-\sqrt{x}} \frac{d}{dx}(-\sqrt{x}) = \frac{-e^{-\sqrt{x}}}{2\sqrt{x}}.$$

f) $f(x) = (x^4 + 1)^{1/3}$

$$\begin{aligned} f'(x) &= \frac{1}{3}(x^4 + 1)^{-2/3} \frac{d}{dx}(x^4 + 1) \\ &= \frac{4x^3}{3(x^4 + 1)^{2/3}}. \end{aligned}$$

g) $f(x) = (2 + \sin x)^x$

$$f(x) = e^{x \ln(2 + \sin x)},$$

so

$$\begin{aligned} f'(x) &= e^{x \ln(2+\sin x)} \frac{d}{dx}(x \ln(2 + \sin x)) \\ &= (2 + \sin x)^x \left(\frac{x \cos x}{2 + \sin x} + \ln(2 + \sin x) \right). \end{aligned}$$

h) $f(x) = \tan^{-1}(x^2)$.

$$\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2},$$

so the chain rule gives

$$f'(x) = \frac{2x}{1+x^4}.$$

2. A missile is launched from the ground with an upward velocity of $200m/s$. Assuming that the acceleration due to gravity is $10m/s^2$, determine:

a) (3 points) The maximum height reached by the missile.

Where $f(t)$ is the height of the missile at time t, its acceleration is $f''(t) = -10$, and $f'(0) = 200$, so $f'(t) = 200 - 10t$. At the maximum height, $f'(t) = 0$, so the maximum height occurs when $t = 20$. Since the initial position is $f(0) = 0$, we have $f(t) = 200t - 5t^2$, so the maximum height is

$$f(20) = 200(20) - 5(20)^2 = 2000m.$$

b) (3 points) The time that elapses before the missile first reaches a height of $1000m$.

We need to find the earliest time t at which $f(t) = 1000$, *i.e.*,

$$200t - 5t^2 = 1000$$

$$5t^2 - 200t + 1000 = 0$$

$$t^2 - 40t + 200 = 0$$

$$t = \frac{1}{2}(40 \pm \sqrt{1600 - 800}) = 20 \pm 10\sqrt{2}.$$

So the earliest time is

$$t = 20 - 10\sqrt{2}\, sec.$$

3 (5 points). Use the tangent line approximation to the function $f(x) = x^3$ near the point $a = 2$ in order to estimate the quantity $(2.03)^3$.

Where $f(x) = x^3$, so that $f'(x) = 3x^2$, we have $f(2) = 8$ and $f'(2) = 12$. So the tangent line approximation gives that, for $x \sim 2$, we have

$$f(x) \sim 8 + 12(x - 2).$$

So

$$(2.03)^3 = f(2.03) \sim 8 + 12(2.03 - 2) = 8.36.$$

4 (5 points). One leg of a right triangle is permanently 3 inches long, while the length of the other leg is increasing at a rate of 2 inches per second. Find the rate of change of the hypotenuse of the triangle at a time that the varying leg is 4 inches long.

Where x is the length of the varying leg and h is the length of the hypotenuse, we have $h^2 = 9 + x^2$, and, currently, $x = 4$ and $\frac{dx}{dt} = 2$. Hence

$$2h\frac{dh}{dt} = \frac{d}{dt}(h^2) = \frac{d}{dt}(9 + x^2) = 2x\frac{dx}{dt}.$$

Now currently $h^2 = 9 + 4^2 = 25$, so $h = 5$. So we get

$$\frac{dh}{dt} = \frac{x}{h}\frac{dx}{dt} = \frac{4}{5}2 = \frac{8}{5}.$$

5 (5 points). Find $\frac{dy}{dx}$ at the point $(1, 2)$ on the curve

$$y^3 + y = x^4 + 9x.$$

Differentiating both sides with respect to x gives

$$3y^2\frac{dy}{dx} + \frac{dy}{dx} = 4x^3 + 9,$$

and then solving for $\frac{dy}{dx}$ shows

$$\frac{dy}{dx} = \frac{4x^2 + 9}{3y^2 + 1}.$$

At the point in question, $x = 1$ and $y = 2$, so

$$\frac{dy}{dt} = \frac{13}{13} = 1.$$

6. Suppose that you know the following two things about some continuous, differentiable function f:

- $f(0) = 1$.
- $f'(x) = 2f(x)$ for every x.

(a) (4 points) Use the chain rule to determine

$$\frac{d}{dx}(\ln(f(x))).$$

(You can assume that $f(x) > 0$ for all x, so that $\ln(f(x))$ makes sense.)

$$\frac{d}{dx}(\ln(f(x))) = \frac{1}{f(x)} f'(x) = \frac{2f(x)}{f(x)}$$
$$= 2.$$

(b) (3 points) In light of part (a), there is only one function that f could possibly be. What is this function, and why is it the only possibility?

The previous part shows that $\ln(f(x))$ has the same derivative (namely, 2) as does the function $g(x) = 2x$. So (by a consequence of the mean value theorem discussed in class) the difference between $ln(f(x))$ and $g(x)$ is some constant D: $\ln(f(x)) = 2x + D$. Now $f(0) = 1$, so $\ln(f(0)) = 0$, so $D = 0$.

Thus $\ln(f(x)) = 2x$, which means that

$$f(x) = e^{2x}.$$