MATH 2250 Exam

April 24, 2009

Name ____

Answer every question on the exam—there is no penalty for guessing. Calculators and similar aids are not allowed. There are a total of **60 points** possible.

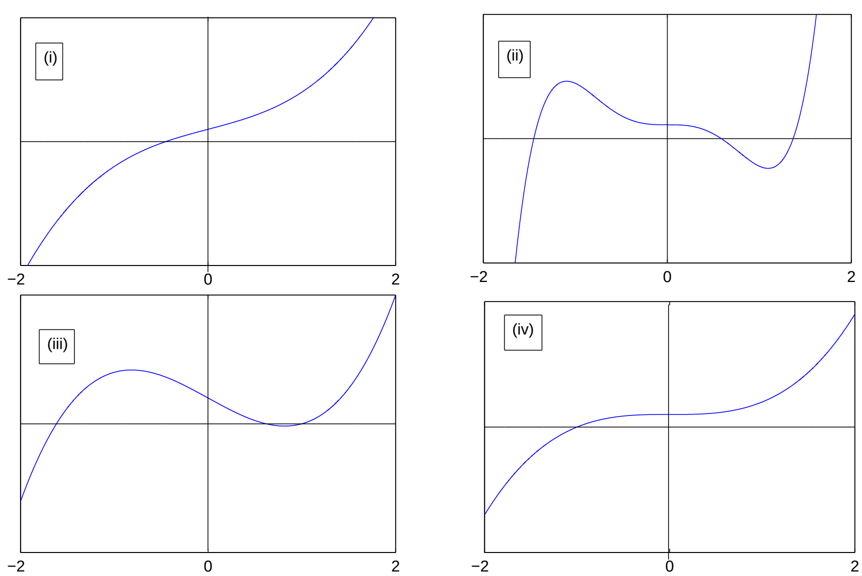
1 (6 points). Below are four graphs of functions (numbered (i)-(iv)), which correspond to the following four functions (with x ranging from -2 to 2). Next to each function, place the number that corresponds to its graph. (You should be able to do this by looking for critical points, local extrema, and inflection points. Incidentally, the vertical scales are different on each graph.)

(a) (iii) $a(x) = x^3 - 2x + 1$

(b) (i) $b(x) = x^3 + 2x + 1$

(c) (ii) $c(x) = 3x^5 - 6x^3 + 1$

(d) (iv) $d(x) = x^3 + 1$



2. Let

$$f(x) = x^5 - 30x^3 + 2.$$

(a) (6 points) Find all inflection points of f.

We have $f'(x) = 5x^4 - 90x^2$, so

$$\begin{aligned} f''(x) &= 20x^3 - 180x = 20x(x^2 - 9) \\ &= 20x(x-3)(x+3). \end{aligned}$$

Thus $f''(x) = 0$ when $x = -3$, $x = 0$, and $x = 3$. Also, $f''(x) < 0$ for $x < 3$, $f''(x) > 0$ for $-3 < x < 0$, $f''(x) < 0$ for $0 < x < 3$, and $f''(x) > 0$ for $x > 3$, so the concavity of f changes at each of these points. Thus the inflection points are at $x = -3, 0, 3$.

(b) (2 points) On the interval $[1, 2]$, is f concave up or concave down?

$f''(1) = 20 - 180 < 0$, and f'' is never 0 for x between 1 and 2, so $f''(x)$ must be negative for all x in $[1, 2]$. Thus f is concave down on this interval.

3 (7 points). You are asked to design a cylindrical can, having total volume 500π cubic centimeters. The material for the side of the can costs 5 cents per square centimeter, while the material for the top and bottom costs 10 cents per square centimeter. What should be the radius and height of the can in order to ensure that the cost to make it is as small as possible?

Call the radius in centimeters of the can r and the height h . Its volume is

$$V = \pi r^2 h = 500\pi,$$

so

$$h = \frac{500\pi}{\pi r^2} = \frac{500}{r^2}.$$

The top and bottom have area each have area πr^2, so the cost to make each separately is $10\pi r^2$ cents (and the total cost for the two is $20\pi r^2$ cents). The side of the can has total area $2\pi rh$, so the cost to make it is $5 \cdot 2\pi rh = 10\pi rh$. So the cost to make the entire can is

$$C = 10\pi rh + 20\pi r^2 = \frac{5000\pi}{r} + 20\pi r^2.$$

Now

$$\frac{dC}{dr} = -\frac{5000\pi}{r^2} + 40\pi r,$$

so $\frac{dC}{dr} = 0$ when $40\pi r = \frac{5000\pi}{r^2}$. Solving for r gives $r^3 = 125$, so $r = 5$. Looking again at $\frac{dC}{dr}$, we see that $\frac{dC}{dr} < 0$ when $r < 5$ and $\frac{dC}{dr} > 0$ when $r > 5$, so $r = 5$ does give a minimum for the cost. Then

$$h = 5000/r^2 = 5000/25 = 20.$$

So the radius of the can should be $5cm$ and its height should be $20cm$.

4. Evaluate the following limits (or show that they don't exist):
(a)(4 points)

$$\lim_{x\to 0}\frac{1-\cos x}{x+\sin x}$$

L'Hôpital's rule gives

$$\lim_{x\to 0}\frac{1-\cos x}{x+\sin x}=\lim_{x\to 0}\frac{\sin x}{1+\cos x}=\frac{0}{2}=0.$$

(b)(4 points)

$$\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)^x$$

Taking logarithms, we see

$$\begin{aligned}\lim_{x\to\infty}\ln\left(1+\frac{1}{x^2}\right)^x &= \lim_{x\to\infty} x\ln(1+1/x^2)\\ &= \lim_{x\to\infty}\frac{\ln(1+1/x^2)}{1/x} = \lim_{x\to\infty}\frac{-\frac{2}{x^3}\frac{1}{1+1/x^2}}{-1/x^2}\\ &= \lim_{x\to\infty}\frac{2}{x}\frac{1}{1+1/x^2} = 0\cdot 1 = 0.\end{aligned}$$

Undoing the logarithm (*i.e.*, using that $a=e^{\ln a}$) then gives

$$\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)^x = e^0 = 1.$$

5 (5 points). Use two steps of Newton's method with $x_1 = 2$ to approximate a solution to the equation

$$x^3 = 3x + 1.$$

(You should find approximations x_2 and x_3, and your final answer should be x_3. You don't have to simplify your answer for x_3).

We wish to approximate a solution to $f(x) = 0$ where $f(x) = x^3 - 3x - 1$. Note that $f'(x) = 3x^2 - 3$. Newton's method says that, if we know x_n, we should set

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Start with $x_1 = 2$. Then

$$x_2 = 2 - \frac{8 - 6 - 1}{6 + 3} = 2 - \frac{1}{9} = \frac{17}{9}.$$

So

$$x_3 = \frac{17}{9} - \frac{(17/9)^3 - 3(17/9) - 1}{3(17/9)^2 - 3}.$$

6 (5 points). A helicopter takes off from the roof of a 100 foot tall building, and its upward velocity at time t seconds after takeoff is $12t^2$ feet per second. How many feet above ground is it 2 seconds after takeoff?

Call its distance above the ground $d(t)$. We're told that $d(0) = 100$, and that $d'(t) = 12t^2$ (velocity is the derivative of displacement). So

$$d(t) = \int d'(t)dt = 4t^3 + C$$

for some constant C. To find C, we have $d(0) = 4(0)^3 + C = C$, while we were given that $d(0) = 100$, so $C = 100$.

This shows that $d(t) = 4t^3 + 100$. When $t = 2$, the helicopter is therefore $d(2) = 132$ feet above ground.

7. (a) (5 points) Using a partition of the interval $[-3, 3]$ into three equal pieces, find the *lower sum* approximation to the area under the graph of the function $f(x) = 10 - x^2$ from $x = -3$ to $x = 3$ (You'll get partial credit if you compute some Riemann sum (like the right endpoint sum) other than the lower sum.)

We partition $[-3, 3]$ into the pieces $[-3, -1], [-1, 1]$, and $[1, 3]$. Each of these pieces has width two.

On $[-3, -1]$, the lowest value of f is $f(-3) = 1$.

On $[-1, 1]$, the lowest value of f is $f(-1) = f(1) = 9$.

On $[1, 3]$, the lowest value of f is $f(3) = 1$.

So the lower sum approximation is

$$1 \cdot 2 + 9 \cdot 2 + 1 \cdot 2 = 22.$$

(b) (4 points) Compute the actual value of the area in part (a) (using the Fundamental Theorem of Calculus).

The area is

$$\int_{-3}^{3} (10 - x^2)dx = \left(10x - \frac{x^3}{3}\right)\Big|_{-3}^{3}$$

$$= (10(3) - 3^3/3) - (10(-3) - (-3)^3/3) = 21 - (-21) = 42.$$

8. Evaluate the following:
(a) (4 points) $\int_0^1 (e^x + 12x^3)dx$

$$\begin{aligned}\int_0^1 (e^x + 12x^3)dx &= \left(e^x + 3x^4\right)\Big|_0^1 \\ &= (e^1 + 3) - (e^0 + 0) = e^1 + 3 - 1 = e + 2.\end{aligned}$$

(b) (4 points) $\int_{-1}^1 \frac{3}{\sqrt{1-x^2}} dx$

$$\int_{-1}^1 \frac{3}{\sqrt{1-x^2}} dx = 3\sin^{-1}(x)|_{-1}^1 = 3\sin^{-1}(1) - 3\sin^{-1}(-1) = 3\pi/2 - (-3\pi/2) = 3\pi.$$

(c) (4 points) $\frac{d}{dx}\left(\int_0^{x^2} \sin(\sqrt{t})dt\right)$

Where $h(x) = \int_0^{x^2} \sin(\sqrt{t})dt$, we're asked to find $\frac{d}{dx}(h(x^2))$. Now part I of the Fundamental Theorem of Calculus gives that $h'(u) = \sin(\sqrt{u})$.

So by the chain rule,

$$\frac{d}{dx}(h(x^2)) = 2xh'(x^2) = 2x \sin\sqrt{x^2} = 2x \sin|x|.$$