Math 2250 Exam 1 September 11, 2006

Name: Solutions (100 points total)

1. (18 points) Find the slope of the graph of $f(x) = \dfrac{2}{x-3}$ at the point $(4, 2)$.

Note: You must use the **definition** of the slope of a curve to solve this problem. You may not use any of the shortcuts from chapter 3.

slope of graph at $x=4$

$$= \lim_{h \to 0} \frac{f(4+h) - f(4)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{2}{(4+h)-3} - \frac{2}{(4)-3}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{2}{1+h} - \frac{2}{1}\right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{2}{1+h} - \frac{2(1+h)}{1+h}\right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{2-2-2h}{1+h}\right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{-2h}{1+h}\right) = \lim_{h \to 0} \frac{-2}{1+h}$$

$$= \frac{-2}{1} = \boxed{-2}$$

2. (32 points) Find the following limits algebraically.

(a) $\lim_{x\to 1}\left(\frac{x-1}{\sqrt{x+3}-2}\cdot\frac{\sqrt{x+3}+2}{\sqrt{x+3}+2}\right) = \lim_{x\to 1}\frac{(x-1)(\sqrt{x+3}+2)}{(x+3)-4}$

$$= \lim_{x\to 1}\frac{\cancel{(x-1)}(\sqrt{x+3}+2)}{\cancel{(x-1)}} = \sqrt{1+3}+2 = 4$$

(b) $\lim_{x\to -3}\frac{x^2+2x-3}{x^2+4x+3} = \lim_{x\to -3}\frac{\cancel{(x+3)}(x-1)}{\cancel{(x+3)}(x+1)}$

$$= \lim_{x\to -3}\frac{x-1}{x+1} = \frac{-3-1}{-3+1} = \frac{-4}{-2} = 2$$

(c) $\lim_{x\to\infty}\frac{\sin(2x)}{x} = 0$, by sandwich Theorem, since

$$\frac{1}{x} \xrightarrow{x\to\infty} 0$$
$$\vee\!\!\mid$$
$$\frac{\sin x}{x} \dashrightarrow 0$$
$$\vee\!\!\mid$$
$$-\frac{1}{x} \xrightarrow{x\to\infty} 0$$

(d) $\lim_{x\to 2^-}\frac{x^2-4}{|x-2|} = \lim_{x\to 2^-}\frac{(x+2)(x-2)}{|x-2|}$

$$= \lim_{x\to 2^-}\frac{(x+2)(x-2)}{-(x-2)} = \lim_{x\to 2^-} -(x+2)$$

$$= -(2+2) = -4$$

(since $|x-2| = -(x-2)$ if $x<2$)

3. (20 points) Let $f(x) = \dfrac{5-x}{x+3}$.
(a) Find the equations of all horizontal and vertical asymptotes.
(b) Sketch the graph of $f(x)$. Make sure to include all of the information from (a) in your sketch.

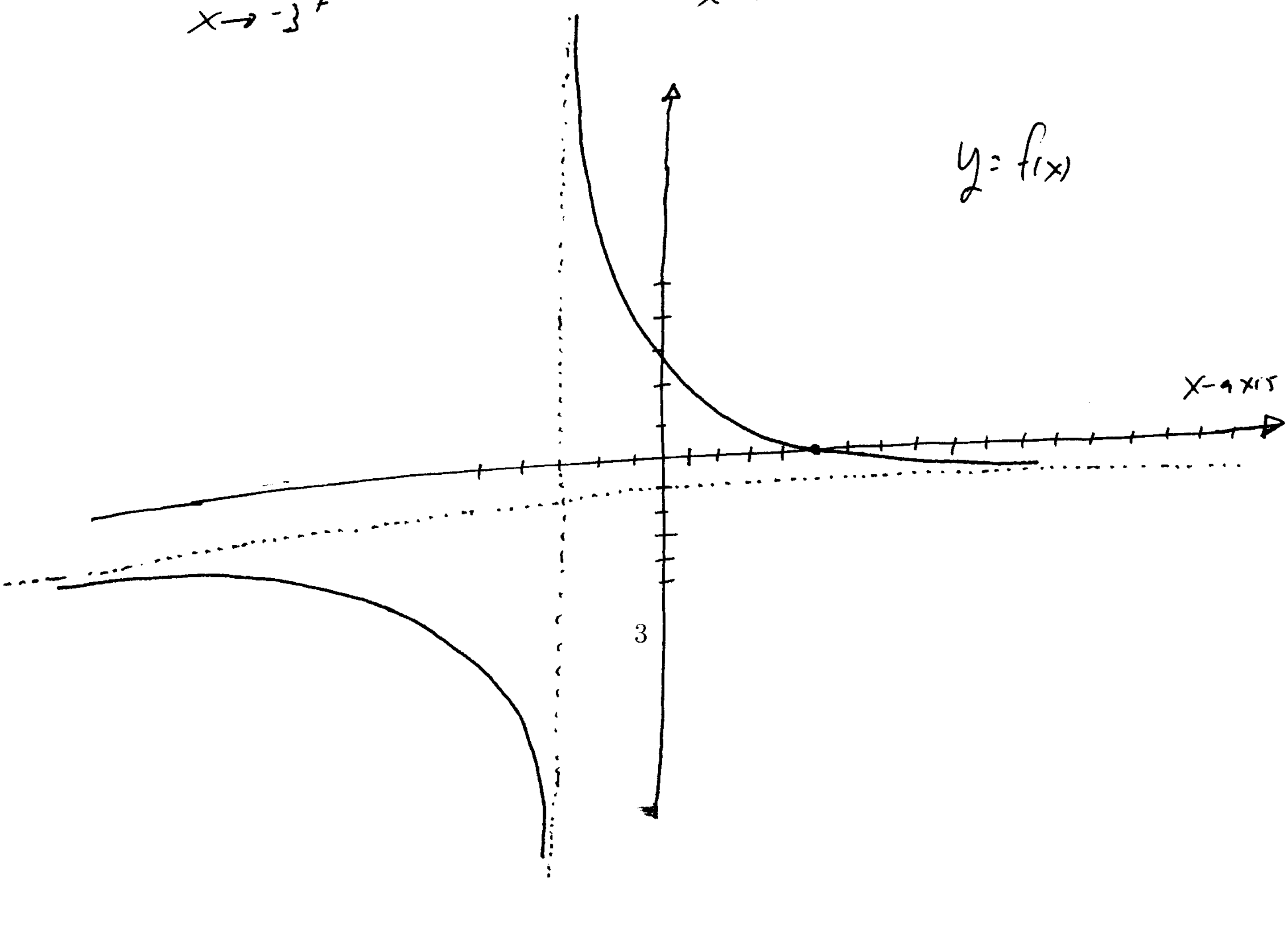
(a) Vertical asymptote : $x = -3$

Horizontal asymptote : $y = -1$, since

$$\lim_{x \to \infty} \frac{5-x}{x+3} = \lim_{x \to -\infty} \frac{5-x}{x+3} = -1$$

(b) Note: for vertical asymptote,

$$\lim_{x \to -3^+} f(x) = \infty \qquad \lim_{x \to -3^-} f(x) = -\infty$$



5. (14 points) Suppose that the inequalities

$$\frac{1}{2} - \frac{x^2}{24} < \frac{1-\cos x}{x^2} < \frac{1}{2}$$

hold for all values of x near 0. What, if anything, does this tell you about $\lim_{x\to 0} \frac{1-\cos x}{x^2}$? Justify your answer.

$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$, by the Sandwich Theorem.

$$\frac{1}{2} - \frac{x^2}{24} < \frac{1-\cos x}{x^2} < \frac{1}{2}$$

$x \to 0$ $\qquad$ $x \to 0$

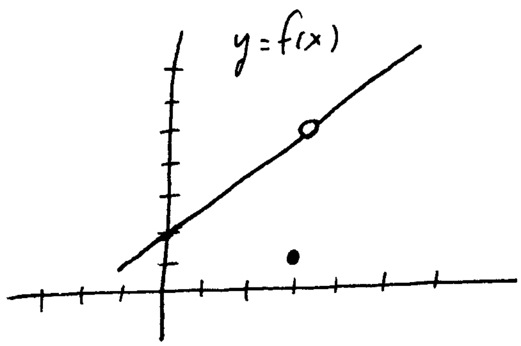
$$\frac{1}{2}$$

6. (16 points) Say whether each of the following is true or false. If true, explain why. If false, give a specific example where the statement does not hold.

(a) If $f(x)$ is continuous at x_0 then $\lim_{x\to x_0} f(x)$ exists. True, by the definition of continuity. $\lim_{x\to x_0} f(x)$ exists is one of the three conditions required for $f(x)$ to be continuous at x_0.

(b) If $\lim_{x\to x_0} f(x)$ exists then $f(x)$ is continuous at x_0. False.

<u>Example</u>. $f(x) = \begin{cases} x+2, & \text{if } x \neq 3 \\ 1, & \text{if } x = 3 \end{cases}$



- $\lim_{x\to 3} f(x)$ exists, but
- $f(x)$ is not continuous at 3.