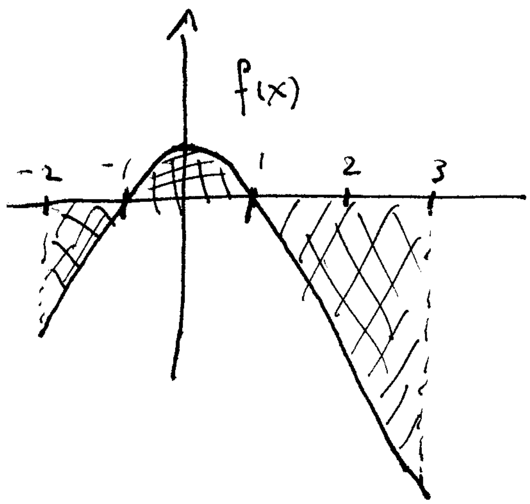
Math 2250 Exam 4 December 4, 2006

Name: Solutions (100 points total)

1. (15 points) Find the total area between $f(x) = -x^2 + 1, \ -2 \le x \le 3$, and the x-axis.



$-x^2+1=0$

$\Rightarrow (1+x)(1-x)=0$

$\Rightarrow x = \pm 1$

$$\text{Area} = -\int_{-2}^{-1}(-x^2+1)\,dx + \int_{-1}^{1}(-x^2+1)\,dx + -\int_{1}^{3}(-x^2+1)\,dx$$

$$= -\left(\frac{-x^3}{3}+x\right)\Big|_{-2}^{-1} + \left(\frac{-x^3}{3}+x\right)\Big|_{-1}^{1} + -\left(\frac{-x^3}{3}+x\right)\Big|_{+1}^{3}$$

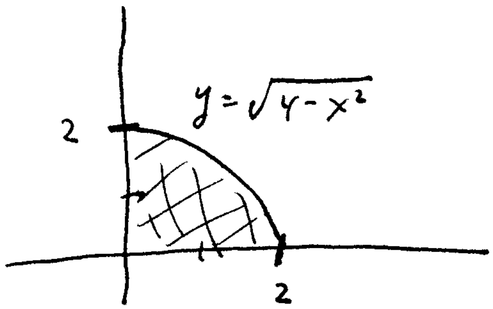
$$= -\left[\left(\frac{-(-1)^3}{3}+-1\right)-\left(\frac{-(-2)^3}{3}+(-2)\right)\right] + \left[\left(\frac{-1^3}{3}+1\right)-\left(\frac{-(-1)^3}{3}+(-1)\right)\right]$$
$$-\left[\left(\frac{-3^3}{3}+3\right)-\left(\frac{-1^3}{3}+1\right)\right]$$

$$= -\left[\left(\frac{1}{3}-1\right)-\left(\frac{8}{3}-2\right)\right] + \left[\left(-\frac{1}{3}+1\right)-\left(\frac{1}{3}-1\right)\right] - \left[\left(-9+3\right)-\left(\frac{-1}{3}+1\right)\right]$$

$$= -\left[-\frac{4}{3}\right] + \left[\frac{4}{3}\right] - \left[\frac{-20}{3}\right] = \boxed{\frac{28}{3}}$$

2. (29 points) Evaluate the following integrals:

(a) $\int_0^2 \sqrt{4 - x^2}\, dx$



$y = \sqrt{4-x^2}$

$y^2 = 4 - x^2, \quad y \geq 0$

$x^2 + y^2 = 4, \quad y \geq 0$

top half of circle,

radius 2, center $(0,0)$

$$\int_0^2 \sqrt{4-x^2}\,dx = \text{Area of shaded region} = \frac{1}{4}(\pi \cdot 2^2) = \boxed{\pi}$$

(b) $\int \tan^2 x \sec^2 x\, dx$

Let $u = \tan x$

$$\frac{du}{dx} = \sec^2 x \Rightarrow du = \sec^2 x\, dx$$

$$\int \tan^2 x \sec^2 x\, dx = \int u^2\, du = \frac{u^3}{3} + C$$

$$= \frac{(\tan x)^3}{3} + C$$

3. (14 points) Find $\dfrac{d}{dx}\displaystyle\int_{\sqrt[3]{x}}^{0.5} \sin\left(t+t^2\right) dt$.

$$= -\frac{d}{dx}\int_{0.5}^{x^{1/3}} \sin(t+t^2)\,dt$$

$$= \sin\left(x^{1/3}+\left(x^{1/3}\right)^2\right)\cdot\left(x^{1/3}\right)'$$

$$= \sin\left(x^{1/3}+x^{2/3}\right)\cdot\frac{1}{3}x^{-2/3} = \frac{\sin\left(\sqrt[3]{x}+\sqrt[3]{x^2}\right)}{3\cdot\sqrt[3]{x^2}}$$

4. (14 points) True or False: $\displaystyle\int x\cos x\,dx = x\sin x+\cos x+C$. **Justify your answer**. Your justification counts for 13 of the 14 points for this problem.

$$\frac{d}{dx}\left(x\sin x+\cos x+C\right)$$

$$= (x)'\sin x + x(\sin x)' - \sin x + 0$$

$$= 1\cdot\sin x + x\cdot\cos x - \sin x = x\cos x,$$

so

$$\int x\cos x = x\sin x+\cos x+C.$$

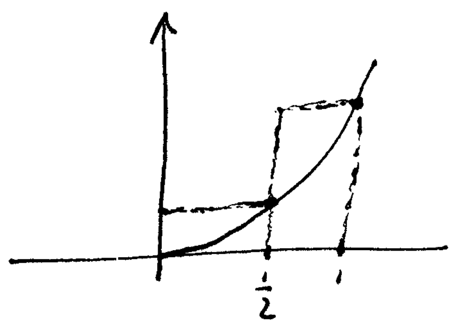
5. (14 points) Find the average value of $f(x) = x^3 + \frac{x}{2}$ on $[-1, 2]$.

$$\text{Avg. Value} = \frac{1}{2-(-1)} \int_{-1}^{2} \left(x^3 + \frac{x}{2}\right) dx$$

$$= \frac{1}{3}\left(\frac{x^4}{4} + \frac{1}{2}\cdot\frac{x^2}{2}\right)\Big|_{-1}^{2} = \frac{1}{12}\left(x^4 + x^2\right)\Big|_{-1}^{2}$$

$$= \left[\frac{1}{12}\left(2^4 + 2^2\right)\right] - \left[\frac{1}{12}\left((-1)^4 + (-1)^2\right)\right] = \frac{20}{12} - \frac{2}{12} = \frac{18}{12} = \boxed{\frac{3}{2}}$$

6. (14 points) Approximate $\int_0^1 x^2\,dx$ by computing the upper sum of $f(x) = x^2$ obtained by dividing the interval $[0, 1]$ into two subintervals of equal length.



$$\int_0^1 x^2 dx \approx (1)^2 \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^2 \cdot \frac{1}{2} = \boxed{\frac{5}{8}}$$