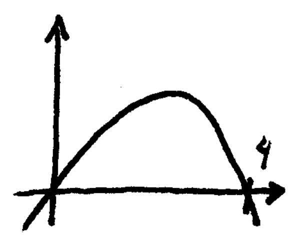
Math 2260 Exam 1 September 25, 2007

Name: Solutions (100 points total)

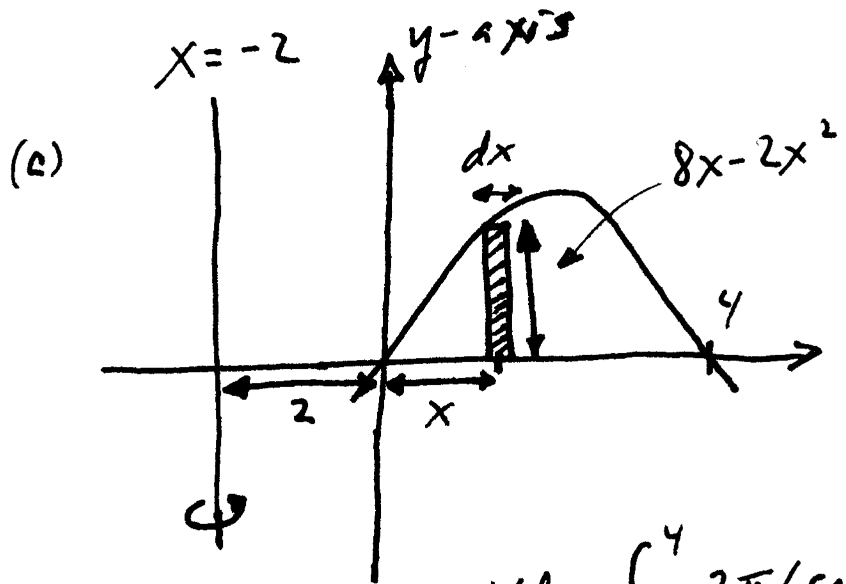
1. (28 points) Let R be the region bounded by the curves $y = 8x - 2x^2$ and $y = 0$. Give definite integrals which compute the volume of the solid obtained by revolving R about (a) the vertical line $x = -2$ and (b) the horizontal line $y = 10$. **Do not evaluate either of these two integrals.**

$y = 8x - 2x^2$

$= 2x(4-x)$



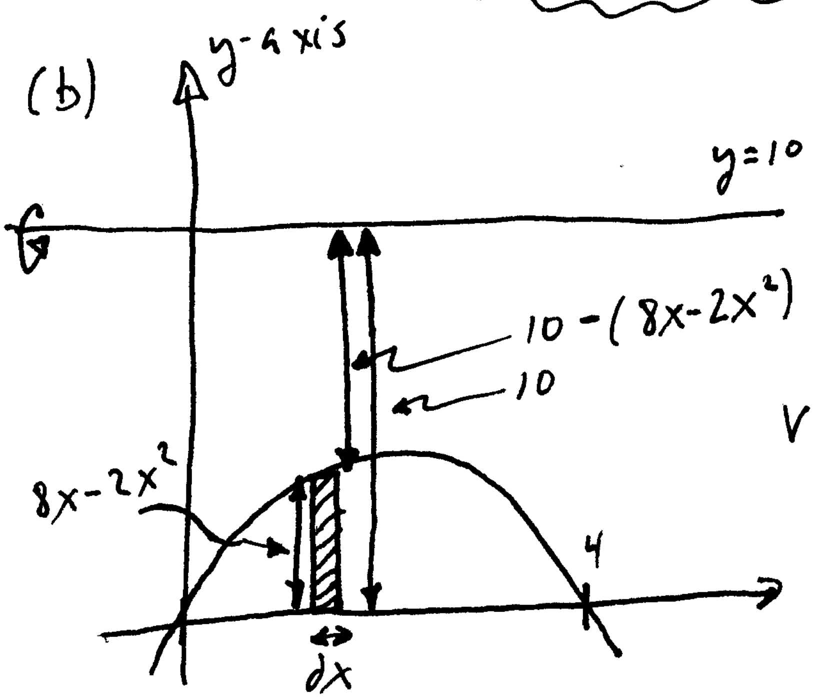
(a)



$$\text{Vol.} = \int_0^4 2\pi(\text{radius})(\text{height})\,dx$$

$$= \int_0^4 2\pi(x+2)(8x-2x^2)\,dx$$

(b)



$$\text{Vol.} = \int_0^4 \left[\pi\left(\begin{matrix}\text{outer}\\ \text{rad.}\end{matrix}\right)^2 - \pi\left(\begin{matrix}\text{inner}\\ \text{rad}\end{matrix}\right)^2\right] dx$$

$$= \int_0^4 \left[\pi(10)^2 - \pi\left(10-(8x-2x^2)\right)^2\right] dx$$

2. (15 points) At the beginning of an experiment, a container holds 100 g of a radioactive substance. After 3 days, 74 g remains. How many days are required for the amount to be reduced to 37 g? Circle one:

(a) 6 (b) 12 (c) $\frac{\ln 0.74}{3\ln 0.37}$ (d) $\frac{3\ln 0.37}{\ln 0.74}$ (e) $\frac{3\ln 0.74}{\ln 0.37}$

let $y(t)$ = Amount of substance at time t (in g)

$$\frac{dy}{dt} = ky, \quad y(0)=100, \; y(3)=74.$$

$$\therefore y = Ae^{kt}$$

$$y(0)=100 \Rightarrow 100 = Ae^{k\cdot 0} = A\cdot 1 \Rightarrow A=100$$

$$\therefore y = 100e^{kt}$$

$$y(3)=74 \Rightarrow 74 = 100e^{k\cdot 3} \Rightarrow k = \frac{1}{3}\ln 0.74$$

$$\therefore \boxed{y = 100\, e^{\frac{1}{3}(\ln 0.74)t}}$$

So ...

$$37 = 100\, e^{\frac{1}{3}(\ln 0.74)t}$$

$$0.37 = e^{\frac{1}{3}(\ln 0.74)t}$$

$$\ln 0.37 = \frac{1}{3}(\ln 0.74)t \Rightarrow \boxed{t = \frac{3\ln 0.37}{\ln 0.74}}$$

$f'(x) = 5x^4 + 2x$

3. (12 points) Consider the curve $y = x^5 + x^2$, $1 \le x \le 5$. Give a definite integral which computes the surface area generated by rotating the curve about (a) the x-axis and (b) the y-axis. **Do not evaluate either of these two integrals.**

(a) Surf. area $= \int_1^5 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$

$$\boxed{= \int_1^5 2\pi(x^5+x^2)\sqrt{1+(5x^4+2x)^2}\,dx}$$

(b) Surf. Area $= \int_1^5 2\pi x\sqrt{1+(f'(x))^2}\,dx$

$$\boxed{= \int_1^5 2\pi x\sqrt{1+(5x^4+2x)^2}\,dx}$$

4. (15 points) Solve the initial value problem $\dfrac{dy}{dx} = e^{3x+2y}$, $y(0) = 5$.

$$\frac{dy}{dx} = e^{3x} \cdot e^{2y}$$

$$\frac{1}{e^{2y}} dy = e^{3x} dx$$

$$\int \frac{1}{e^{2y}} dy = \int e^{3x} dx$$

$$\int e^{-2y} dy = \int e^{3x} dx$$

$$-\frac{1}{2} e^{-2y} = \frac{1}{3} e^{3x} + C$$

$$e^{-2y} = -\frac{2}{3} e^{3x} + C$$

$$\ln(e^{-2y}) = \ln\left(-\frac{2}{3} e^{3x} + C\right)$$

$$-2y = \ln\left(-\frac{2}{3} e^{3x} + C\right)$$

$$\therefore y = -\frac{1}{2} \ln\left(-\frac{2}{3} e^{3x} + C\right)$$

<u>Now use $y(0) = 5$ to find C:</u>

$$e^{-2(5)} = -\frac{2}{3} e^{3(0)} + C$$

$$e^{-10} = -\frac{2}{3} + C$$

$$C = e^{-10} + \frac{2}{3}$$

$$\therefore \boxed{y = -\frac{1}{2} \ln\left(-\frac{2}{3} e^{3x} + e^{-10} + \frac{2}{3}\right)}$$

5. (15 points) Find the length of the curve $x^2 = y^3$ from $(0,0)$ to $(1,1)$.

$$x^2 = y^3$$
$$x = y^{3/2}$$
$$\frac{dx}{dy} = \frac{3}{2} y^{1/2}$$

$$L = \int_0^1 \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_0^1 \sqrt{1+\left(\frac{3}{2} y^{1/2}\right)^2}\, dy$$

$$= \int_0^1 \sqrt{\underbrace{1+\frac{9}{4}y}_{u}}\ \underbrace{dy}_{\frac{4}{9}du}$$

$$= \int_{y=0}^{1} \frac{4}{9}\cdot\sqrt{u}\, du$$

$$= \int_{y=0}^{1} \frac{4}{9}\cdot u^{1/2}\, du$$

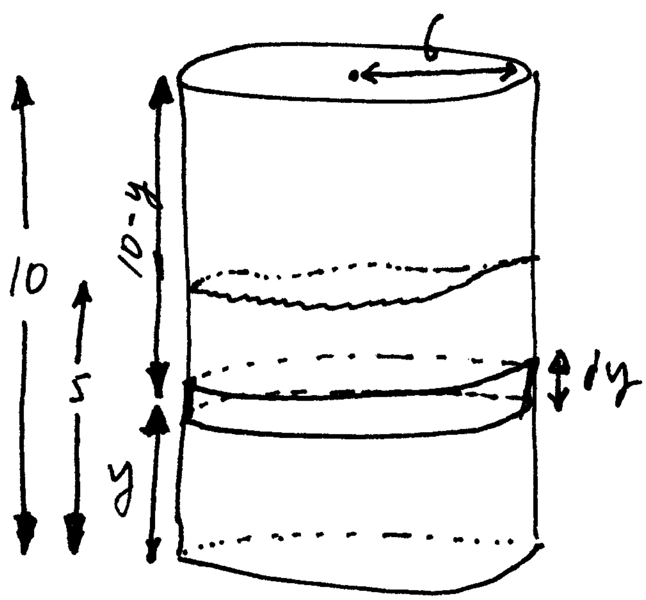
$$= \frac{4}{9}\cdot \frac{u^{3/2}}{3/2}\Bigg|_{y=0}^{1}$$

$$= \frac{4}{9}\cdot\frac{\left(1+\frac{9}{4}y\right)^{3/2}}{3/2}\Bigg|_{y=0}^{1}$$

$$= \frac{8}{27}\left(1+\frac{9}{4}y\right)^{3/2}\Bigg|_0^1$$

$$= \boxed{\frac{8}{27}\left(\frac{13}{4}\right)^{3/2} - \frac{8}{27}(1)^{3/2}}$$

6. (15 points) A cylindrical barrel of height 10 feet and radius 6 feet is half full of a liquid whose density is 40 pounds per cubic foot. Give a definite integral which computes the amount of work required to pump all of the liquid over the top of the barrel. **Do not evaluate the integral.**



Volume of representative slab

$= \pi \cdot 6^2 \cdot dy \ ft^3$

Weight of representative slab

$= \left(\pi \cdot 6^2 \cdot dy \ ft^3\right)\left(40 \ \frac{lbs.}{ft^3}\right)$

$= \pi \cdot 6^2 \cdot 40 \, dy \ lbs.$

Work to lift representative slab to top

$= \underbrace{\left(\pi \cdot 6^2 \cdot 40 \, dy \ lbs.\right)}_{\text{force}} \times \underbrace{\left(10 - y \ ft.\right)}_{\text{dist.}}$

$= \pi \cdot 6^2 \cdot 40 \cdot (10-y) \, dy \quad ft.-lbs.$

$\therefore$ Total work $= \lim_{dy \to 0} \sum \pi \cdot 6^2 \cdot 40 \cdot (10-y) dy = \boxed{\int_0^5 \pi \cdot 6^2 \, 40 \cdot (10-y) dy}$