Math 2260 Exam 3 November 20, 2007

Name: Solutions (100 points total)

1. (20 points) Determine whether the following series converge or diverge. Justify your answers and state all convergence tests used.

(a) $\sum_{n=3}^{\infty} \frac{n^2 + 5\sqrt{n} + 7}{n^4 - n^2 - 3}$

$$\left.\begin{array}{l} \bullet \displaystyle\lim_{n\to\infty} \frac{\left(\frac{n^2+5\sqrt{n}+7}{n^4-n^2-3}\right)}{\left(\frac{1}{n^2}\right)} = 1 \\ \\ \bullet \displaystyle\sum_{n=3}^{\infty} \frac{1}{n^2} \text{ Converges (p-series, } p=2) \end{array}\right\} \Longrightarrow \sum \frac{n^2+5\sqrt{n}+7}{n^4-n^2-3}$$

converges, by the Limit Comparison Test

(b) $\sum_{n=1}^{\infty} \left(\frac{1}{2} + \frac{1}{n^2}\right)$

$$\frac{1}{2} + \frac{1}{n^2} \xrightarrow{n\to\infty} \frac{1}{2} \neq 0$$

$\therefore$ Series diverges by the n-th term test for divergence.

$$\frac{4^n+1}{3^{2n}} = \frac{4^n+1}{(3^2)^n} = \frac{4^n+1}{9^n} = \frac{4^n}{9^n} + \frac{1}{9^n} = \left(\frac{4}{9}\right)^n + \left(\frac{1}{9}\right)^n$$

2. (20 points) Determine whether the following series converge or diverge. **If a series converges, find its sum**. Justify your answers and state all convergence tests used.

(a) $\sum_{n=2}^{\infty} \frac{4^n+1}{3^{2n}}$ $= \sum_{n=2}^{\infty} \left(\frac{4}{9}\right)^n + \left(\frac{1}{9}\right)^n$

$$= \sum_{n=2}^{\infty} \left(\frac{4}{9}\right)^n + \sum_{n=2}^{\infty} \left(\frac{1}{9}\right)^n$$

Converges by geometric series to

$$\frac{\left(\frac{4}{9}\right)^2}{1-4/9} + \frac{\left(\frac{1}{9}\right)^2}{1-1/9}$$

(b) $\sum_{n=1}^{\infty} \left(\frac{n}{2n+3} - \frac{n+1}{2(n+1)+3}\right)$ (Telescoping Series)

$$S_1 = \frac{1}{2\cdot 1+3} - \frac{2}{2\cdot 2+3}$$

$$S_2 = \left(\frac{1}{2\cdot 1+3} - \cancel{\frac{2}{2\cdot 2+3}}\right) + \left(\cancel{\frac{2}{2\cdot 2+3}} - \frac{3}{2\cdot 3+3}\right) = \frac{1}{2\cdot 1+3} - \frac{2}{2\cdot 3+3}$$

$\vdots$

$$S_n = \left(\frac{1}{2\cdot 1+3} - \cancel{\frac{2}{2\cdot 2+3}}\right) \cancel{+ \cdots +} \left(\cancel{\frac{n}{2\cdot n+3}} - \frac{n+1}{2(n+1)+3}\right)$$

$$= \frac{1}{2\cdot 1+3} - \frac{n+1}{2(n+1)+3}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{2\cdot 1+3} - \frac{n+1}{2n+5} = \frac{1}{2\cdot 1+3} - \frac{1}{2} = \frac{1}{5} - \frac{1}{2}$$

$$= \frac{-3}{10}$$

$\therefore$ series converges to $\frac{-3}{10}$

3. (20 points) Does $\sum_{n=3}^{\infty} \frac{(-1)^n}{n \ln n}$ converge absolutely, converge conditionally, or diverge? Justify your answer, and state all convergence tests used.

Does series converge absolutely?

$$\sum_{n=3}^{\infty} \left| \frac{(-1)^n}{n \ln n} \right| = \sum_{n=3}^{\infty} \frac{1}{n \ln n} .$$ ← use integral test

$$\int_3^{\infty} \frac{1}{x \ln x}\, dx = \lim_{b\to\infty} \left(\int_3^b \frac{1}{\underbrace{\ln x}_{u}} \cdot \overbrace{\frac{1}{x}\,dx}^{du} \right)$$

$$= \lim_{b\to\infty} \left(\ln(\ln x) \Big|_3^b \right)$$

$$= \lim_{b\to\infty} \ln(\ln(b)) - \ln(\ln(3)) = \infty$$

$\therefore \sum \left| \frac{(-1)^n}{n \ln n} \right|$ does not converge

$\therefore \sum \frac{(-1)^n}{n \ln n}$ does not converge absolutely.

Does series converge?

- $\frac{1}{n \ln n} > 0$, all $n \geq 3$

- $\frac{1}{(n+1)\ln(n+1)} \leq \frac{1}{n \ln n}$, $n \geq 3$

- $\frac{1}{n \ln n} \xrightarrow{n \to \infty} 0$

$\therefore \sum_{n=3}^{\infty} (-1)^n \cdot \frac{1}{n \ln n}$ converges, by Alternating series test.

Conclusion:
Since $\sum_{n=3}^{\infty} (-1)^n \cdot \frac{1}{n \ln n}$ converges, but does not converge absolutely, it converges conditionally.

4. (20 points) Find the interval of convergence of $\sum_{n=1}^{\infty} \frac{3^n \sqrt{n}\,(x-4)^n}{(n+2)!}$.

$$\lim_{n\to\infty} \left| \frac{\dfrac{3^{n+1}\sqrt{n+1}\,(x-4)^{n+1}}{((n+1)+2)!}}{\dfrac{3^n\sqrt{n}\,(x-4)^n}{(n+2)!}} \right| = \lim_{n\to\infty} \left| \overset{3}{\frac{3^{n+1}}{3^n}} \cdot \frac{\sqrt{n+1}}{\sqrt{n}} \overset{\frac{1}{n+3}}{\frac{(n+2)!}{(n+3)!}} \overset{x-4}{\frac{(x-4)^{n+1}}{(x-4)^n}} \right|$$

$$= \lim_{n\to\infty} \left(3 \frac{\sqrt{n+1}}{\sqrt{n}} \cdot \frac{1}{n+3} \cdot |x-4| \right) = 3|x-4| \lim_{n\to\infty} \frac{\sqrt{n+1}}{(n+3)\sqrt{n}}$$

$$= 3|x-4| \cdot 0 = 0 < 1, \text{ for all } x$$

$\therefore$ Series converges for all x, by ratio test

$\therefore$ IOC $= (-\infty, \infty)$.

5. (20 points) Let $\sum_{i=1}^{\infty} a_i$ be a series. Define $s_n = \sum_{i=1}^{n} a_i$. Indicate whether each of the following statements is true or false (no explanations required).

(a) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ must converge. FALSE

(b) If $\lim_{n\to\infty} s_n = 2$, then $\sum_{n=1}^{\infty} a_n$ must converge. TRUE

(c) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, then $\sum_{n=1}^{\infty} a_n$ must converge. FALSE

(d) If $\sum_{n=1}^{\infty} |a_n|$ diverges, then $\sum_{n=1}^{\infty} a_n$ must diverge. FALSE