Math 2260 Exam 1 February 13, 2007

Name: Solutions (100 points total)

1. (20 points) A sample of a radioactive substance decays at a rate proportional to the amount present. Suppose that after two hours the amount of the substance decreases to one-third of its initial amount. How long will it take for the substance to be reduced to one-tenth of its initial amount?

① $\frac{dy}{dt} = ky$, $y(0) = y_0$, $y(2) = \frac{1}{3}y_0$

② $\therefore y(t) = Ae^{kt}$

③ Use $y(0) = y_0$ to find A

$(y_0) = Ae^{k(0)}$

$y_0 = A \cdot 1$, so $A = y_0$

$\therefore y = y_0 e^{kt}$

④ Use $y(2) = \frac{1}{3}y_0$ to find k

$\left(\frac{1}{3}y_0\right) = y_0 e^{k \cdot 2}$

$\frac{1}{3} = e^{k \cdot 2}$

$\ln\left(\frac{1}{3}\right) = k \cdot 2$, so $k = \frac{1}{2}\ln\left(\frac{1}{3}\right)$

$\therefore y = y_0 e^{\frac{1}{2}\ln\left(\frac{1}{3}\right)t}$

⑤ Find t for which $y(t) = \frac{1}{10}y_0$

$\left(\frac{1}{10}y_0\right) = y_0 e^{\frac{1}{2}\ln\left(\frac{1}{3}\right)t}$

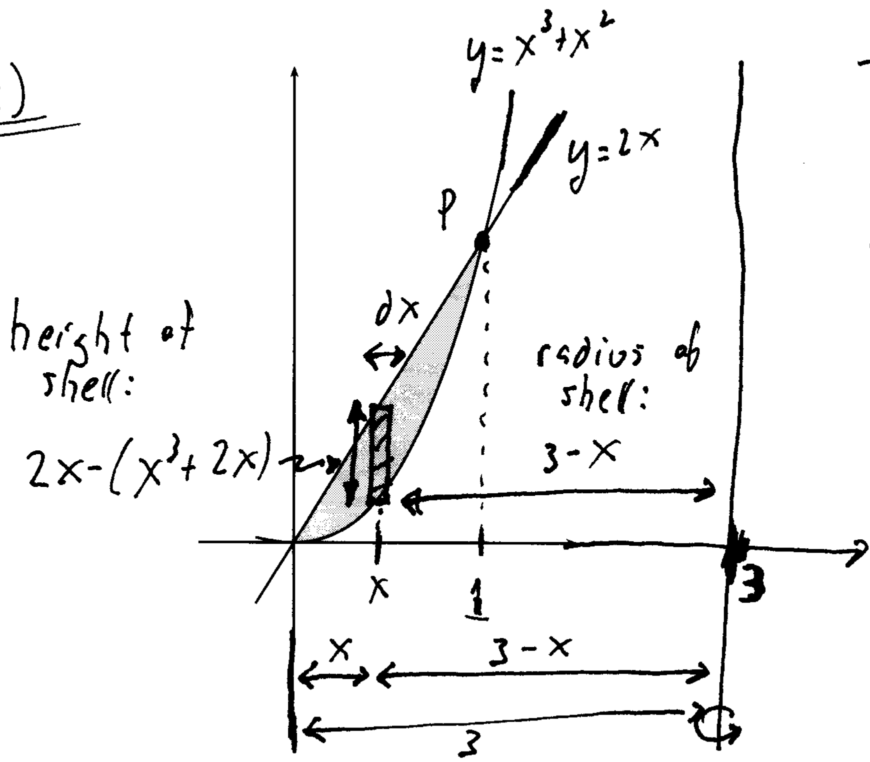
$\frac{1}{10} = e^{\frac{1}{2}\ln\left(\frac{1}{3}\right)t}$

$\ln\left(\frac{1}{10}\right) = \frac{1}{2}\ln\left(\frac{1}{3}\right)t$

$$\boxed{t = \frac{\ln\left(\frac{1}{10}\right)}{\frac{1}{2}\ln\left(\frac{1}{3}\right)}}$$

2. (35 points) The region R in the first quadrant bounded by the curve $y = x^3 + x^2$ and the line $y = 2x$ is sketched below. Give definite integrals which compute the volume of the solid generated by revolving R about (a) the vertical line $x = 3$ and (b) the horizontal line $y = -1$. Do not evaluate either of these two integrals.

2(a)



to find P

$x^3 + x^2 = 2x$

$x^3 + x^2 - 2x = 0$

$x(x^2 + x - 2) = 0$

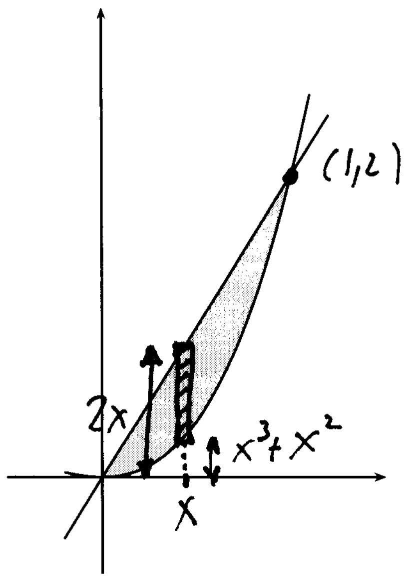
$x(x+2)(x-1) = 0$

$x = 0$ or $x = -2$ or $x = 1$ P

$$\text{(a) Volume} = \int_0^1 2\pi \underbrace{(3-x)}_{\text{radius of shell}} \underbrace{[2x - (x^3 + 2x)]}_{\text{height of shell}} \, dx$$

~~2. (35 points) The region R in the first quadrant bounded by the curve $y = x^3 + x^2$ and the line $y = 2x$ is sketched below. Give definite integrals which compute the volume of the solid generated by revolving R about (a) the vertical line $x = 3$ and (b) the horizontal line $y = -1$. Do not evaluate either of these two integrals.~~

2(b)



outer radius: $2x$

inner radius: $x^3 + x^2$

$$\text{Volume} = \int_0^1 \left[\pi (2x)^2 - \pi (x^3 + x^2)^2 \right] dx$$

3. (20 points) Consider the curve $y = 3\sqrt{x}$, $1 \le x \le 2$. Give integrals which compute (a) the length of the curve and (b) the surface area generated by rotating the curve about the x-axis. Do not evaluate either of the two integrals.

$$y = 3\sqrt{x} = 3x^{1/2}$$

$$\frac{dy}{dx} = \frac{3}{2}x^{-1/2} = \frac{3}{2\sqrt{x}}$$

(a) Length $= \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \boxed{\int_1^2 \sqrt{1 + \left(\frac{3}{2\sqrt{x}}\right)^2}\, dx}$

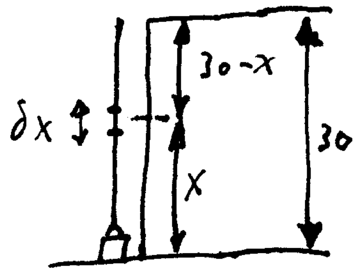
(b) Surface area $= \int_1^2 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$$\boxed{= \int_1^2 2\pi \cdot 3\sqrt{x} \sqrt{1 + \left(\frac{3}{2\sqrt{x}}\right)^2}\, dx}$$

4. (25 points) A 6 pound bucket with water is lifted from the ground to the top of a 30-ft building by pulling in rope which weighs 0.1-lb/ft at a constant speed. The bucket is leaking. It starts with 3 pounds of water, leaks water at a constant rate, and finishes draining just as it reaches the top. How much work is spent lifting the rope, bucket, and water? Hint: you must determine the amount of work required to lift the rope, bucket, and water separately, and then add these three numbers.

<u>Bucket</u>: Work = Force × dist. = 6 lbs. × 30 ft. = 180 ft.-lbs.

<u>rope</u>:



$$\text{Work} = \int_0^{30} \overbrace{(30-x)}^{\text{dist.}} \cdot \overbrace{(0.1\,dx)}^{\text{Force}}$$

$$= \int_0^{30} (3 - .1x)\,dx = \left(3x - .1\cdot\frac{x^2}{2}\right)\Bigg|_0^{30}$$

$$= x\left(3 - \frac{x}{20}\right)\Bigg|_0^{30} = 30\left(\frac{3}{2}\right) = 45$$

<u>Water</u>: weight of water x ft. from ground: $3 - .1x$
(so that weight is 3 when $x = 0$, weight is 0 when $x = 30$).

$$\text{Work} = \int_0^{30} \underbrace{(3 - .1x)}_{\text{Force}} \cdot \underbrace{dx}_{\text{dist.}} = 45 \text{ (same integral as above).}$$

$\therefore$ total work $= 180 + 45 + 45 = \boxed{270 \text{ ft.-lbs.}}$