Math 2260 Exam 2 March 20, 2007

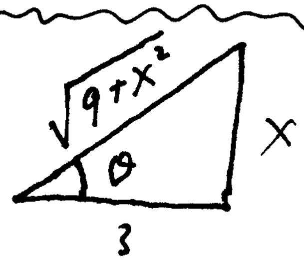
Name: Solutions (100 points total)

1. (85 points) Evaluate the following integrals.

(a) $\int \frac{dx}{\sqrt{9+x^2}} = \int \frac{3\sec^2\theta \, d\theta}{3\sec\theta} = \int \sec\theta \, d\theta$

$= \ln|\sec\theta + \tan\theta| + C$

$= \ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| + C$



$x = 3\tan\theta$

$\Rightarrow dx = 3\sec^2\theta \, d\theta$

$\sqrt{9+x^2} = 3\sec\theta$

(b) $\int \frac{e^x \, dx}{e^{2x} + 3e^x + 2} = \int \frac{du}{u^2+3u+2} = \int \frac{1}{(u+2)(u+1)} \, du$

[redacted]

Let $u = e^x$

$du = e^x dx$

$= \int \left(\frac{-1}{u+2} + \frac{1}{u+1} \right) du$

$= -\ln|u+2| + \ln|u+1| + C$

partial fraction expansion

$= -\ln|e^x+2| + \ln|e^x+1| + C$

(c) $\int \cos^5 x \, dx$

$$= \int \cos^4 x \cos x \, dx = \int \left(1 - \overset{u}{(\sin x)^2}\right)^2 \overset{du}{\cos x \, dx}$$

$$= \int (1-u^2)^2 \, du$$

$$= \int (1 - 2u^2 + u^4) \, du$$

$$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C$$

$$= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C$$

(d) $\int \frac{\ln x}{x^2} \, dx$

$$= \int \overset{u}{\ln x} \, \overset{dv}{\frac{1}{x^2} dx} = (\ln x)\left(\frac{-1}{x}\right) - \int \frac{-1}{x} \cdot \frac{1}{x} \, dx$$

$u = \ln x \qquad v = \frac{-1}{x}$

$du = \frac{1}{x} dx \qquad dv = \frac{1}{x^2} dx$

$$= -\frac{\ln x}{x} + \int \frac{1}{x^2} dx$$

$$= \frac{-\ln x}{x} + \frac{-1}{x} + C$$

(e) $\int \frac{10x^2}{(x-1)(x^2+9)}\,dx$

$$\frac{10x^2}{(x-1)(x^2+9)} = \frac{1}{x-1} + \frac{9x+9}{x^2+9}$$

$$= \int \left(\frac{1}{x-1} + \frac{9x+9}{x^2+9} \right) dx$$

$$= \int \frac{1}{x-1}\,dx + 9\int \frac{x}{x^2+9}\,dx + \int \frac{9}{x^2+9}\,dx$$

$$= \ln|x-1| + 9\int \frac{\overset{\frac{1}{2}du}{x\,dx}}{\underset{u}{x^2+9}} + \int \frac{1}{1+\left(\frac{x}{3}\right)^2}\,dx$$

$$= \ln|x-1| + \frac{9}{2}\ln|x^2+9| + 3\tan^{-1}\left(\frac{x}{3}\right) + C$$

2. (15 points) Determine whether $\int_{-2}^{1} \frac{1}{x^2}\, dx$ converges or diverges. Justify your answer.

$$\int_{-2}^{1} \frac{1}{x^2}\, dx = \int_{-2}^{0} \frac{1}{x^2}\, dx + \int_{0}^{1} \frac{1}{x^2}\, dx$$

- $$\int_{0}^{1} \frac{1}{x^2} dx = \lim_{a \to 0^+} \left(\int_a^1 \frac{1}{x^2} dx \right)$$

$$= \lim_{a \to 0^+} \left(-\frac{1}{x} \Big|_a^1 \right) = \lim_{a \to 0^+} \left(-1 + \frac{1}{a} \right) = \infty$$

$\therefore \int_0^1 \frac{1}{x^2} dx$ diverges, which implies that $\int_{-2}^{1} \frac{1}{x^2} dx$ diverges.

(Note: It can be checked that $\int_{-2}^{0} \frac{1}{x^2} dx$ diverges as well. However, this is unnecessary. If either $\int_0^1 \frac{1}{x^2} dx$ or $\int_{-2}^{0} \frac{1}{x^2} dx$ diverges, then so does $\int_{-2}^{1} \frac{1}{x^2} dx$.)